The Mathematical Work of Ted Kaczynski

Ted Kaczynski

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Introduction by Jørgen Veisdal

The Mathematics of Ted Kaczynski

Disclaimer: As should be fairly evident, this essay is in no way meant to glorify Ted Kaczynski. Rather, it was written with two goals in mind: 1. To orient to reality some of the myths of Kaczynski's "genius" and 2. To illustrate yet another example of a mathematician whose abstract endeavours ultimately defeated him.

Before terrorist Theodore John Kaczynski (1942-) began sending mail-bombs to faculty members at various American universities, he had a promising career in mathematics. In particular, between 1964–69, he published a total of six single-authored research papers in renowned mathematical journals, including *The American Mathematical Monthly* and *Proceedings of the American Mathematical Society.*

The young Kaczynski did work in analysis, specifically geometric function theory in the narrow subfield of boundary values of continuous functions. The purpose of this article is to give an introduction to this work.

Education (1958–67)

Kaczynski grew up in Illinois, where he attended Sherman Elementary School and Evergreen Park Central Junior High school. At the age of 10 years old, his IQ was evaluated to be 167, and so he skipped the sixth grade (Chicago Tribune, 2017), an event later described as pivotal to his development (Chase, 2004):

> "Previously he had socialized with his peers and was even a leader, but after skipping ahead he felt he did not fit in with the older children and was bullied."

Harvard University (1958–62)

Kaczynski entered Harvard University in 1958 at the age of 16 years old. A mathematical prodigy since he was a child, he was described by other undergraduates as *"shy"*, *"quiet"* and *"a loner"* who *"never talked to anyone"* (Song, 2012):

> "He would just rush through the suite, go into his room, and slam the door [...] When we would go into his room there would be piles of books and uneaten sandwiches that would make the place smell"

His personality notwithstanding, Kaczynski's talent was however still recognized among his Harvard peers, one of which in 2012 stated:

> "It's just an opinion — but Ted was brilliant [...]. He could have become one of the greatest mathematicians in the country"

Kaczynski graduated Harvard with a B.A. in mathematics in 1962. When he graduated, his GPA was 3.12, scoring B's in the History of Science, Humanities and Math, C in History and A's in Anthropology and Scandinavian (Stampfl, 2006).

University of Michigan (1962–67)

With an IQ of 167, Kaczynski had been expected to perform better at Harvard. After graduating, he applied to the University of California at Berkeley, The University of Chicago and the University of Michigan. Although accepted at all three, he ended up choosing Michigan because the university offered him an annual grant of $2,310 and a teaching post. The "darling of the math department", he would graduate from the University of Michigan in 1964 with a M.Sc. in mathematics and markedly improved grades — 12 A's and five B's, which he himself later attributed to the standing of the university:

> "[My] memories of the University of Michigan are **not** pleasant [...] The fact that I not only passed my courses (except one physics course) but got quite a few A's shows how wretchedly low the standards were at Michigan"

Nonetheless, as the story goes, while there once a professor named George Piranian told his students — including Kaczynski — about an unsolved problem in boundary functions. Weeks later, Kaczynski came to his office with a 100-page correct, handwritten proof. Kaczynski graduated with a Ph.D. in mathematics in 1967. His dissertation, entitled simply *"Boundary Functions"* regarded the same topic as his proof of Piranian's problem. His doctoral committee consisted of professors Allen L. Shields, Peter L. Duren, Donald J. Livingstone, Maxwell O. Reade, Chia-Shun Yin. Every professor approved it. His supervisor Shields later called his dissertation

> "The best I have ever directed"

An additional testament to its quality was it being awarded the Sumner Myers Prize for the best mathematics thesis of the university, accompanying a prize of $100 and a plaque in the East Quad Residence Hall entrance listing his accomplishment. Of the complexity (or perhaps narrow implications) of his dissertation, one of the members of his dissertation committee, Maxwell Reade, said

> "I would guess that maybe 10 or 12 men in the country understood or appreciated it"

Another, Peter Duren, stated

> "He was really an unusual student"

Kaczynski at UCB in 1967 (Photo: Wikimedia Commons)

University of California, Berkeley (1967–69)

In late 1967, at 25 years old Kaczynski was hired as the youngest-ever assistant professor of mathematics at the University of California at Berkeley. There, he taught undergraduate courses in geometry and calculus, although with mediocre success. His student evaluations suggest that he was not particularly well-liked because he taught *"straight from the textbook and refused to answer questions"*.

He resigned on June 30th, 1969 without explanation.

Work (1964–69)

Wedderburn's Theorem

Kaczynski's only published paper relating to topics other than boundary functions was his first journal paper, written before he started his Ph.D. It is entitled:

- **Kaczynski, T.J. (1964).** "Another proof of Wedderburn's theorem". *The American Mathematical Monthly* 71(6), pp. 652–653.

The paper concerned a 1905 result of Joseph H. M. Wedderburn that every finite skew field is commutative. His paper provided a group-theoretic proof of the theorem, which had previously been proved at least seven times.

Boundary Functions

Kaczynski's Ph.D. dissertation concerned boundary values of continuous functions and was entitled, simply

- **Kaczynski, T.J. (1967).** *Boundary Functions.* Ann Arbor: University of Michigan.

> Let H denote the set of all points in the Euclidean plane having positive y-coordinate, and let X denote the x-axis. If p is a point of X, then by an arc at p we mean a simple arc γ, having one endpoint at p, such that γ = {p} ⊠ H. Let f be a function mapping H into the Riemann sphere.
>
> **Boundary Functions** By a boundary function for f we mean a function φ defined on a set E ⊠ X such that for each p ⊠ E there exists an arc γ at p for whichlim (s p, s ⊠ γ) f(z) = φ(p)

Kaczynski's dissertation begins by re-proving a theorem of J. E. McMillan which states that if f(H) is a a continuous function mapping H into the Riemann sphere, the the set of curvilinear convergence of F (the largest set on which a boundary function for f can be defined) is of a certain type. This proof also shows that if A is a set of the same type in X, then there exists a bounded continuous complex-valued function in H having A as its set of curvilinear convergence. The dissertation contains two additional new proofs related to boundary functions, and a list of problems for future research. Of the results, Professor Donald Rung later stated:

> What Kaczynski did, greatly simplified, was determine the general rules for the properties of sets of points of curvilinear convergence. Some of those rules were not the sort of thing even a mathematician would expect.

Kaczynski would publish five journal papers related to the work from his dissertation between 1965–69:

- **Kaczynski, T.J. (1965)**. "Boundary functions for functions defined in a disk". *Journal of Mathematics and Mechanics.* 14(4), pp. 589–612.
- **Kaczynski, T.J. (1966)**. "On a boundary property of continuous functions". *Michigan Math. J.* 13, pp. 313–320.
- **Kaczynski, T.J. (1969)**. "The set of curvilinear convergence of a continuous function defined in the interior of a cube". *Proceedings of the American Mathematical Society* 23(2), pp. 323–327.
- **Kaczynski, T.J. (1969)**. "Boundary functions and sets of curvilinear convergence for continuous functions". *Transactions of the American Mathematical Society.* 141, pp. 107–125.
- **Kaczynski, T.J. (1969)**. "Boundary functions for bounded harmonic functions". *Transactions of the American Mathematical Society.* 137, pp. 203–209.

The Distributivity Problem

The only other trace of Kaczynski in a mathematical journal is two notes in the American Monthly in 1964 and 65:

- **Kaczynski, T.J. (1964).** "Distributivity and $(-1)x = -x$ (Advanced Problem 5210)". *The American Mathematical Monthly.* 71(6), pp. 689.
- **Kaczynski, T.J. (1965)**. "Distributivity and $(-1)x = -x$ (Advanced Problem 5210, with Solution by Bilyeu, R.G.)". *The American Mathematical Monthly* 72(6), pp. 677–678.

In the first note, Kaczynski proposes the following problem, concerning group theory:

> Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying:1. K is an abelian group under addition, 2. K - {0} is a group under multiplication, and 3. x(y+z) = xy + xz for all x,y,z ⊠ K.Suppose that for some n, 0=1+1+1...+1 (n times). Prove that, for all x ⊠ K, (-1)x = -x.

In the second note, the solution to the problem is — somewhat dismissively — provided by R. G. Bilyeu:

> The last part of the hypothesis is unnecessary. If z denotes -1, then z+z+zz = z(1+1+z) = z, so zz = 1. Now z(x+zx) = zx+x = x+zx, so either x+zx = 0 or z = 1. In either case zx = -x.

Conclusion

Theodore J. Kaczynski was a very promising young undergraduate, graduate and post-graduate student in the 1960s. His work — although pertaining to vary narrow topics — was undoubtedly, technically, first rate.

As is the case however, elegance or complexity do not themselves raise the importance of problems, achievements or for that matter, mathematicians. As expressed by his fellow graduate student Professor Peter Rosenthal in a 1996 Toronto Star article (after Kaczynski was charged):

> [The] topic was only of interest to a very small group of mathematicians and does not appear to have broader implications; thus, his work had little impact. Kaczynski might have quit mathematics because he was discouraged by the resultant lack of recognition.

In another 1996 article, in the Los Angeles Times article, Professor Donald Rung similarly expressed:

> "The field that Kaczynski worked in doesn't really exist today [...]. He probably would have gone on to some other area if he were to stay in mathematics," Rung said. "As you can imagine, there are not a thousand theorems to be proved about this stuff."

An Advanced Explanation of His Breakthrough by Lara Pudwell

Original PDF: Digit Reversal Without Apology.pdf

Digit Reversal Without Apology

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In A Mathematician's Apology[1] G. H. Hardy states, "8712 and 9801 are the only four-figure numbers which are integral multiples of their reversals"; and, he further comments that "this is not a serious theorem, as it is not capable of any significant generalization."

However, Hardy's comment may have been short-sighted. In 1966, A. Sutcliffe[2] expanded this obscure fact about reversals. Instead of restricting his study to base 10 integers and their reversals, Sutcliffe generalized the problem to study all integer solutions of

$k(ahn^h + a_{h-1}n^{h-}1 + \bullet\bullet\bullet + a1\ n + a_0) = a_0n^h + a1n^{h-1} + \bullet\bullet\bullet + a_{h-1}n + ah$

with n $>2,\ 1 < k < n,\ 0 < ai < n - 1$ for all i, $a_0 = 0$, $a_h = 0$. We shall refer to such an integer $a0...ah$ as an $(h + 1)$-digit solution for n and write $k(a_h,\ a_{h-1},\ ...,\ a_1,\ a_0)_n = (a_0,\ a_1,\ ...,\ a_{h-1},\ a_h)_n$. For example, 8712 and 9801 are 4-digit solutions in base $n = 10$ for $k = 4$ and $k = 9$ respectively. After characterizing all 2-digit solutions for fixed n and generating parametric solutions for higher digit solutions, Sutcliffe left the following open question: Is there any base n for which there is a 3-digit solution but no 2-digit solution?

Two years later T. J. Kaczynski(1)[3] answered Sutcliffe's question in the negative. His elegant proof showed that if there exists a 3-digit solution for n, then deleting the middle digit gives a 2-digit solution for n. Together with Sutcliffe's work, this proved that there exists a 2-digit solution for n if and only if there exists a 3-digit solution for n.

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[1] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat, Acad, Sci. U. S. A.*> 4 (1955) 379-382.

[2] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

[3] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17 (1931) 283-295.

(1) Better known for other work.

Given the nice correspondence between 2- and 3-digit solutions described by Sutcliffe and Kaczynski, it is natural to ask if there exists such a correspondence for higher digit solutions. In this paper, we will explore the relationship between 4- and 5-digit solutions. Unfortunately, there is not a bijection between these solutions, but there is a nice family of 4- and 5- digit solutions which have a natural one-to-one correspondence.

A second extension of Sutcliffe and Kaczynski's results is to ask, "Is there any value of n for which there is a 5-digit solution but no 4-digit solution?" We will answer this question in the negative; and, furthermore, we will show that there exist 4- and 5-digit solutions for every n >3.

An attempt at generalization

In the case of 3-digit solutions, Kaczynski proved that if n + 1 is prime and k(a, b, c)n = (c, b, a)n is a 3-digit solution for n, then k(a, c)n = (c, a)n is a 2-digit solution. Thus, we consider the following:

Question 1. Let k(a, b, c, d, e)n = (e, d, c, b, a)n be a 5-digit solution for n. If n + 1 is prime, then is $k(a, b, d, e)_n = (e, d, b, a)_n$ a 4-digit solution for n?

First, following Kaczynski, let p = n + 1. We have

$k(an^4 + bn^3 + cn^2 + dn + e) = en^4 + dn^3 + cn^2 + bn + a.$ (1)

Reducing this equation modulo p, we obtain

$k(a - b + c - d + e) = e - d + c - b + a = a - b + c - d + e \mod p.$

Thus, $(k - 1)(a - b + c - d + e) = 0 \mod p$, and

$p \mid (k - 1)(a - b + c - d + e)$. (2)

Ifp $\mid (k-1)$, then $k-1 > p$, which is impossible because $k < n$. Therefore, $p \mid (a - b + c - d + e)$. But $-2p < -2n < a - b + c - d + e < 3n < 3p$, so there are four possibilities:

(i) a - b + c - d + e = -p,

(ii) a - b + c - d + e = 0,

(iii) a - b + c - d + e = p, (iv) a - b + c - d + e = 2p.

Write $a - b + c - d + e = fp$, where f G {—1,0,1,2}. Substituting c = -a + b + d - e + fp into equation 1 gives:

$k[n^2 (n^2 - 1)a + n^2(n + 1)b + fpn^2 + n(n + 1)d - (n^2 - 1)e]$

$= n^2(n^2 - 1)e + n^2(n + 1)d + fpn^2 + n(n + 1)b - (n^2 - 1)a.$

After substituting for p, dividing by n + 1, and rearranging, one sees that $k [an^3 + (b - a + f)n^2 + (d - e)n + e] = en^3 + (d - e + f)n^2 + (b - a)n + a$. Indeed, this is a 4-digit solution for n if f = 0, $b - a > 0$, and $d - e > 0$, but not necessarily a 4-digit solution of the form conjectured in Question 1.

As in Kaczynski's proof for 2- and 3-digit solutions, it would be ideal if three of the four possible values for f lead to contradictions and the fourth leads to a "nice" pairing of 4- and 5-digit solutions. Unlike Kaczynski, we now have the added advantage of exploring these cases with computer programs such as Maple. Experimental evidence suggests that the cases f = —1 and f = 2 are impossible. The cases f = 0 and f = 1 are discussed below.

A counterexample

Unfortunately, Kaczynski's proof does not completely generalize to higher digit solutions. Most 5-digit solutions do, in fact, yield 4-digit solutions in the manner described in Question 1, but for sufficiently large n there are examples where (a, b, c, d, e)n is a 5-digit solution but (a, b, d, e)n is not a 4-digit solution.

A computer search shows that the smallest such counterexamples appear when n = 22:

7(2, 8, 3, 13, 16)22 = (16, 13, 3, 8, 2)22, 3(2, 16, 11, 5, 8)22 = (8, 5, 11, 16, 2)22.

However, there is no integer k for which k(2, 8, 13, 16)22 = (16, 13, 8, 2)22 or k(2, 16, 5, 8)22 = (8, 5, 16, 2)22. Note that -2 + 8 + 13 - 16 = 3 and -2 + 16 + 5 - 8 = 11; that is, both of these counterexamples to Question 1 occur when f = 0. The next smallest counterexamples are

3(3, 22, 15, 7, 11)30 = (11, 7, 15, 22, 3)30, 8(2, 13, 8, 16, 9)30 = (9, 16, 8, 13, 2)30,

which occur when f = 0 and n = 30.

A family of 4- and 5-digit solutions

Although Kaczynski's proof does not generalize entirely, there exists a family of 5-digit solutions when f = 1 that has a nice structure.

Theorem 1. Fix n *>2 and a* > 0. Then

k(a, a - 1, n - 1, n - a - 1, n - a)n = (n - a, n - a - 1, n - 1, a - 1, a)n

is a 5-digit solution for n if and only if a | (n - a).

Proof. We have

(n - a)n⁴ + (n - a - 1)n³ + (n - 1)n² + (a - 1)n + a $an^4 + (a - 1)n^3 + (n - 1)n^2 +$ (n - a - 1)n + (n - a)

(n - a)(n⁴ + n³ - n - 1) n - a $a(n^4 + n^3 - n - 1)$ a ·

and the result is clear. ⊠

Notice that

(-a+ (a - 1)) + ((n - a - 1) - (n - a)) +p = -1 + -1 + (n+ 1) = n - 1.

That is, this family of solutions occurs when f = 1. Moreover, this family follows the pattern described in Question 1; that is, for each 5-digit solution described in Theorem 1, deleting its middle digit gives a 4-digit solution.

Theorem 2. If

k(a, a - 1,n - 1, n - a - 1,n - a)n = (n - a,n - a - 1,n - 1, a - 1, a)n

is a 5-digit solution for n, then

k(a, a - 1, n - a - 1,n - a)$_n$ = (n - a,n - a - 1, a - 1, a)$_n$

is a 4-digit solution for n.

Proof. By Theorem 1, n-a G N. Now

$(n - a)n^3 + (n - a - 1)n^2 + (a - 1)n + a$

$an^3 + (a - 1)n^2 + (n - a - 1)n + (n - a)$

$(n - a)(n^3 + n^2 - n - 1)$ n - a

$a(n^3 + n^2 - n - 1)$ a ·

⊠

These 4-digit solutions were first described by Klosinski and Smolarski[4] in 1969, but their relationship to 5-digit solutions was not made explicit before now.

It is also interesting to note that 9801 and 8712, the two integers in Hardy's discussion of reversals, are included in this family of solutions.

We conclude with the following corollary.

Corollary 1. *There is a 4-digit solution and a 5-digit solution for every* $n > 3$.

Proof. Let $a = 1$ in the statements of Theorem 1 and Theorem 2 above. ⊠

Some open questions

We have shown that there is no n for which there is a 5-digit solution but no 4-digit solution. More specifically, we know that there are 4- and 5-digit solutions for every n > 3.

Although Kaczynski's proof does not generalize directly to 4- and 5-digit solutions, it does bring to light several questions about the structure of solutions to the digit reversal problem.

First, it would be interesting to completely characterize 4- and 5-digit solutions for n. Namely,

1. All known counterexamples to Question 1 occur when $f = 0$. Are there counterexamples for which f 6= 0? Is there a parameterization for all such counterexamples?

2. Theorems 1 and 2 exhibit a family of 4- and 5-digit solutions for $f = 1$ with a particularly nice structure. To date, no other 4- or 5-digit solutions are known for $f = 1$. Do such solutions exist?

More generally,

3. Solutions to the digit reversal problem have not been explicitly characterized for more than 5 digits. Do there exist analogous results to Theorems 1 and 2 for higher digit solutions?

A Maple package for exploring these questions is available from the author's web page at http://www.math.rutgers.edu/~lpudwell/maple.html.

[4] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

Acknowledgment

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Ted's Work as a Michigan PhD Student

1. June 1964 - Another Proof of Wedderburn's Theorem

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Kaczynski, T.J. 1964. Another proof of Wedderburn's theorem. *Am. Math. Month.* 71:652-653.

Another Proof of Wedderburn's Theorem

Author(s): T. J. Kaczynski

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ANOTHER PROOF OF WEDDERBURN'S THEOREM

T. J. Kaczynski, Evergreen Park, Illinois

In 1905 Wedderburn proved that every finite skew field is commutative. At least seven proofs of this theorem (not counting the present one) are known. See[1],[2],[3] (Part Two, p. 206 and Exercise 4 on p. 219),[4] (two proofs), and[5]. Unlike these proofs, the

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[1] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat, Acad, Sci. U. S. A.*> 4 (1955) 379-382.

[2] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

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[4] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

[5] H. Tietze, Uber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.,* 5 (1919), 284-291.

proof to be given here is group-theoretic, in the sense that the only non-group-theoretic concepts employed are of an elementary nature.

Lemma. *Let q be a prime. Then the congruence* $Z^2+r^2= — 1$ (mod *q) has a solution t, r with t^O* (mod *q).*

Proof. If —1 is a quadratic residue, take r = 0 and choose *t* appropriately. Assume — 1 is a nonresidue. Then any nonresidue can be written in the form — s^2 (mod *q)* with s^O. If t^2+r^2 is ever a nonresidue for some *t,* r, set t^2+r^2 s— s^2, and we have $(/5^{\sim 1})^2 + (r5"^1)^2 = — 1$. (Throughout this note, x^{-1} denotes that integer for which xx^{-1} = l (mod *q).)* On the other hand, if t^2+r^2 is always a residue, then the sum of any two residues is a residue, so —l=g—1 = 1 + 14- • • • + 1 is a residue, contradicting our assumption.

Proof of the theorem. Let *F* be our finite skew field, E* its multiplicative group. Let 5 be any Sylow subgroup of F*, of order, say, p^a. Choose an element *g* of order *p* in the center of 5. If some *h^S* generates a subgroup of order *p* different from that generated by g, then *g* and *h* generate a commutative field containing more than *p* roots of the equation x^p=l, an impossibility. Thus 5 contains only one subgroup of order *p* and hence is either a cyclic or a generalized quaternion group ([6] p. 189).

If S is a generalized quaternion group, then 5 contains a quaternion subgroup generated by two elements *a* and &, both of order 4, where *ba — a~^b.* Now a^2 generates a commutative field in which the only roots of the equation x^2 — 1 or (x+l)(x—1) =0 are ±1, so since $(a^2)^2$ = l, we have

(1) a^2 = - 1.

Hence *a^ — a^2 — —a_t* so

(2) *ba* = — *ab.*

Similarly,

(3) 5^2 = - 1.

Taking *q* — characteristic of *F* (#-l = 0), choose *t* and *r* as specified in the lemma. Using relations (1), (2), (3), we have

(/ + ra + 5)(r^2 + 1 + *rta + tb)* = r($/^2$ + r^2 + l)a + ($/^2$ + r^2 + 1)5 = 0.

One of the factors on the left must be 0, so for some numbers *u, v, w, u* 0 (mod g), we have w+^a+^5 = 0, or *b= -u^wa-u^w.* So *b* commutes with *a,* a contradiction. We conclude that 5 is not a generalized quaternion group, so 5 is cyclic.

Thus every Sylow subgroup of F* is cyclic, and F* is solvable ([7], pp. 181— 182). Let *Z* be the center of F* and assume Z^F*. Then *F*/Z* is solvable, and its Sylow subgroups are cyclic. Let *A/Z* (with ZC^4) be a minimal normal subgroup of *F*/Z.* *A/Z* is an elementary abelian group of order (F prime), so since the Sylow subgroups of

[6] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17 (1931) 283-295.

[7] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be any element of F*, y any element of A. Since A is normal, $xyx^{\sim x}$ ^A, and (l+x)y = z(l+x) for some zf^A. An easy manipulation shows that y — z — zx — xy = $(z$ — $xyx^{\sim l})x$.

If y — z = z — xyx^{-1} = 0, then $y = z = xyx^{\sim 1}$, so x and y commute. Otherwise, x= $(z$ — $xyx^{\sim 1}$'$)^{\sim 1}(y$ — $z)$. But A is abelian, and 2, y, xyx^{-1}(E^4, so x commutes with y. Thus we have proven that A is contained in the center of F*, a contradiction.

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A NOTE ON PRODUCT SYSTEMS OF SETS OF NATURAL NUMBERS

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In this note, we apply a slight twist to a trick exploited about twelve years ago by J. C. E. Dekker (*[2*]), our purpose being to expose a couple of elementary facts about nonempty, countable "product systems" of infinite sets of natural numbers which are, at the same time, "finite symmetric difference systems." We proceed in terms of the following definitions.

Definition. *By a product system of subsets of N (N the natural numbers), we mean a collection of subsets of N which contains, along with any two of its members, their intersection.*

2. 1964 - Distributivity and (−1)x = −x (Advanced Problem 5210)

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Kaczynski, T.J. (1964). "Distributivity and (−1)x = −x (Advanced Problem 5210)". *The American Mathematical Monthly.* 71(6), pp. 689.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. *Barlaz, Rutgers - The State University, New Brunswick, N.J. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before December 31, 1964*
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5210. *Proposed by T.* J. *Kaczynski, Evergreen Park, Illinois*

Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying:

1. K is an abelian group under addition,

2. K - {O} is a group under multiplication, and

3. $x(y + z) == xy + xz$ for all x,y,z EK.

Suppose that for some *n,* `0 = 1 + 1 + ... + 1 ( <em>n</em> times). Prove that, for all <em>x</em> eK, (-l)x =` *-x.*

3. 1964 - Distributivity and (−1)x = −x (Advanced Problem 5210, with Solution by Bilyeu, R.G.)

Original PDF: 3. Distributivity and (−1)x = −x (Advanced Problem 5210, with Solution by Bilyeu, R.G.).pdf

Kaczynski, T.J. (1965). "Distributivity and (−1)x = −x (Advanced Problem 5210, with Solution by Bilyeu, R.G.)". *The American Mathematical Monthly* 72(6), pp. 677–678.

Distributivity and (-1 *)x == -x*

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1. K is an abelian group under addition,

2. K - {O} is a group under multiplication, and

3. *x(y + z) == xy + xz* for all *x,y,z* EK.

Suppose that for some *n,* 0 = 1 + 1 + ... + 1 (n times). Prove that, for all x eK, (-l)x = *-x.*

Solution by R. G. Bilyeu, North Texas State University. The last part of the hypothesis is unnecessary. If *z* denotes -1, then *z + z + z z* = z (1 + 1 + z) = *z,* so *z(?)* = 1. Now z(x + zx) = *zx + x* = x + zx, so either x + zx = 0 or *z* = 1. In either case z(?) = *-x.*

Also solved by Carol Avelsgaard, Richard Bourgin, Robert Bowen, Joel Brawley, Jr., F. P. Callahan, M. M. Chawla (India), R. A. Cunninghame-Green (England), M. J. DeLeon, M. Edelstein, N. J. Fine, Harvey Friedman, Anton Glaser, M. G. Greening (Australia), A. G. Heinicke, Sidney Heller, G. A. Heuer, Stephen Hoffman, K. G. Johnson, A. J. Karson, Max Klicker, Kwangil Koh, C. C. Lindner, C. R. MacCluer, H. F. Mattson, C. J. Maxson, R. V. Moddy, Jose Morgado (Brazil), W. L. Owen, Jr., P. R. Parthasarathy (India), Harsh Pittie, Kenneth Rogers, Toru Saito (Japan), Camilio Schmidt, Leonard Shapiro, Frank A. Smith, George Van Zwalenberg, W. C. Waterhouse, Kenneth Yanosko, and the proposer.

4. 1965 - Boundary Functions for Functions Defined in a Disk

Original PDF: 4. 1965 Boundary Functions for Functions Defined in a Disk.pdf

Kaczynski, T.J. 1965. Boundary functions for functions defined in a disk. *J. Math. and Mech.* 14(4):589-612.

MR0176080 Kaczynski, T. J. Boundary functions for function defined in a disk. J. Math. Mech. 14 1965 589.612. (Reviewer: C. Tanaka) 30.62

Explanation by John D. Bullough

Let D denote the unit disk |z| < 1, C its boundary, and let f(z) be any function that is defined in D and takes its values in some metric space S. Then a boundary function for f is a function t on C such that for every x (C there exists an arc v at x with

lim f(z) = t(x).

z -> x

z (v

The author proves several theorems on boundary functions in the following four cases: (1) f(z) a homeomorphism of D onto D, (2) f(z) a continuous function, (3) f(z) a Baire function and (4) f(z) a measurable function. These theorems include answers to two questions raised by Bagemihl and Piranian.

Theorem 1 states that if f(z) is a homeomorphism of D onto D, then there exists a countable set N such that t|C - N is continuous.

In the case of continuous functions, one needs some definitions. Let S and T be metric spaces. f is said to be of Baire class 1(S, T) if and only if (i) domain f = S, (ii) range f (T and (iii) there exists a sequence {f(n)} of continuous functions, each mapping S into T, such that f(n) -> f pointwise on S. g is of honorary Baire class 2(S, T) if and only if (i) domain g = S, (ii) range g (T and (iii) there exists a function f of Baire class 1(S, T) and a countable set N such that f|S - N = g|S - N. Using these defnitions, Theorems 2 and 3 read as follows. Theorem 2: Let f be a continuous real-valued function in D and let t be a finite-valued boundary function for f. Then t is of honorary Baire class 2(C, R), where R is the set of real numbers. Theorem 3:

Let f be a continuous function mapping D into the Riemann sphere S and let t be a boundary function for f. Then t is of honorary Baire class 2(C, S).

In the cases of Baire functions and measurable functions, for the sake of convenience consider the open upper half-plane D^0: I(z) > 0, and its boundary C^0: I(z) = 0, instead of D and C, respectively. Theorem 4 states that if f is a real-valued function of Baire class a > 1 in D^0, and t is a finite-valued boundary function, then t is of Baire class a + 1. As an immediate consequence of Theorem 4, one has Theorem 5: Let f be a real-valued Borel-measurable function in D^0 and let t be a finite-valued boundary function for f; then t is Borel-measurable.

Next, the author proves that for an arbitrary function t on C^0, there exists a function f on D^0 such that f(z) = 0 almost everywhere and t is a boundary function for f. The paper concludes with some remarks concerning extensions of these theorems into three dimensions.

Article by Ted

Boundary Functions for Functions Defined in a DisB

T. J. KACZYNSKI

Communicated *by* F. Bagemihl

1. Introduction

Throughout this paper *D* will denote the open unit disk (in two-dimensional Euclidean space) and *C* will denote its boundary, the unit circle. Bagemihl and Piranian[1] have introduced the following definition.

Definition. If *x* e C, an *arc at x is &* simple arc *y* having one endpoint at *x* such that *y — {x} C D.* Let / be any function that is defined in *D* and takes its values in some metric space S. Then a *boundary junction* for f is a function <*p* on *C* such that for every *x* e *C* there exists an arc *y* at *x* with

lim *f (z) = <p(x).*

The purpose of this paper is to prove several theorems concerning boundary functions. These theorems include answers to two questions raised in[2] (see Problem 1 and the conjecture on p. 202).

The set of real numbers will be denoted by *R,* W-dimensional Euclidean space will be denoted by R^N, and the Riemann sphere will be denoted by 2. Points in R^N will be written in the form *{x*$_x$ *, x*$_2$ *} • • • , x*$_N$*)* rather than *(x*$_t$ *, x*$_2$ *, • • • , x*$_N$*)* (to avoid confusion with open intervals of real numbers in the case *N —* 2). Whenever

[1] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

[2] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

we speak of real-valued functions we mean finite-valued functions, and whenever we speak of increasing functions we refer to weakly increasing (nondecreasing) functions. The abbreviations "l.u.b." and "g.l.b." stand for "least upper bound" and "greatest lower bound" respectively. Finally, it should be noted that our definition of the Baire classes is slightly unconventional (see p. 6 and p.14) in that we consider Baire class *a* to include Baire class ft for every $ft < a$.

2. Boundary functions for homeomorphisms.

Definition. If *E* C *D,* let acc *(E)* denote the set of all points on *C* which are accessible by arcs in *E.*

[1]1 would like to thank Professor G. Piranian for his encouragement.

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Journal of Mathematics and Mechanics, Vol. 14, No. 4 (1965).

Lemma 1. Let A be an arcwise connected subset of D and let B be a connected subset of D. Suppose that A B —</> . Then acc (A) *and B have at most two points in common.*

Proof. Assume that p_r , p_2 , p_3 are three distinct points of acc (A) A *B* and derive a contradiction. Let 7* be an arc joining *pi* to a point *qi z A,* with *{Pi} A (i* = 1, 2, 3). Let *y* be an arc in *A* joining and q_2 . Putting

, 7_2 and 7 together, we obtain an arc T joining pj to p_2, with r — $\{p_r, p_2\}$ *C A. We* can assume F is a simple arc, for if r is not simple, and p_2 can be joined by some simple arc F^z £ T (see[3]). Let L_x , L_2 be the two open arcs of *C* determined by the pair of points p_r , p_2 . We may assume, by symmetry, that p_3 *z* L_r . According to[4] (Theorem 11.8, p. 119), *D* — T has two components *U*i and U_2 , the boundary of *Ui* being L_x U P and the boundary of U_2 being z_2u r.

Let 7' be an arc in *A* joining q_3 *to a* point *q z* T *A.* Putting 73 and 7' together, we obtain an arc 5 joining p_3 to *q.* Starting at p_3 and proceeding along 5, let *r* be the first point of T that we reach. Let A be the subarc of 5 with endpoints at p_3 and *r.* Clearly, A — {p_3} *GZ A.* We can assume (according to[5]) that A is a simple arc.

Since p_3 *z* L_x , p_3 is not in U_2 . Since

A - $\{p_3$, r$\}$ £ *D* - T = *Ui* U_2 ,

A — *{ps , r}* must have a point in U_x . But A — $\{p_3$, r$\}$ is connected, so A - $\{p_3$, r$\}$ *C Ui.* Hence A is a cross cut of U_r. Let M_r ,M_2*/oq* the two open subarcs of *Li* with endpoints p_t , p_3 and p_2 , p_3 respectively. Let Pi , r_2 be the two closed subarcs of T

[3] H. Tietze, Uber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.,* 5 (1919), 284-291.

[4] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

[5] H. Tietze, Uber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.,* 5 (1919), 284-291.

with endpoints p_r , r and p_2 , r respectively. According to[6] (Theorem 11.8, p. 119), U_r — A has two components *Vi* and V_2, the boundary of *Vi* being *kJ* r_x kJ A and the boundary of V_2 being M_2 *kJ* r_2 kJ A.

Since P U A C 2I, *Vi^J* V_2*\J* U_2. Recall that p_3 *4* U_2 . It follows that since p_3 *z B, B* has a point in common with *Vi^J* V_2 . But *B* is connected, so *B £ Vi yj* V_2. We see that p_r *$* V_2, and therefore that *B Vi* 4=</> (because *Pi z B).* Hence *B £* , so p_2 *z* W . But, since the boundary of V_t is M_x

I\ *kJ* A, p_2 *Vi* . This contradiction proves the lemma.

Lemma 2. There exists a countable family 8 *of open disks such that every open set U Q* R^2 *can be written in the form U —* S_n *, where* S_n*z § and* S_n *£ U.*

Proof. Let $\{p_n\}$ be a countable dense subset of R^2, and let 8 be the family of all open disks of rational radius having some p_n as center. 8 is clearly countable. If *U* is an open set it is easy to show that for each *x z U* there exists an S_x *z* 8 with *x z* S_x *£* S_x *£ U.* Obviously

u = *\J* s_x.

xt.U

Theorem 1. Let f be a homeomorphism of D onto D, and let <p be a boundary function for f. Then there exists a countable set N such that <p $/_C$*-^ is continuous.*

Prooj. Take an arbitrary aS £ S. It is easily shown that *DPS* and *D* — *S* are both connected, so f^{-1}(D *(P 8)* and — *8)* are both connected. Given x_0 e *C,* let *y* be any arc at x_Q . If

x_Q *$* acc *($f^{-1}$(D iP* aS)),

then we can choose points on *y* arbitrarily close to x_Q which are not in $f^{\sim 1}$ *(D (P 8), so*

x_0 *£ D - r\D CPS') = r\D* -aS).

This shows that

(1) *C Q* acc aS)) kJ *r*~*\D* -aS).

Let

F = acc *(f-\D fP* aS)) H *f*~*\D* -aS).

By Lemma 1, *F* contains at most two points, and from (1) we see that acc *(rXD S)) = FU (C - r\b* - aS)).

Thus we have shown that for each aS £ S we can write

acc *(T\D* aS)) = *F$ kJ* G_s ,

where F_s is finite and G_s is open (relative to C).

For any arc *y* at a point *x* on C, the *cluster set C(f,* t) of *f* along *y* is defined by

C(/, t) = {w £ 2?[2] kJ { oo } | there exists a sequence *{z_n} QyPD* such that z_n —> *x* and f (z_n) —> w}.

Let

E = {x £ C | there exist arcs y_x, y_2 at *x* such that C(/, *yj IP C(J,* y_2) =</> }•

[6] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

A theorem of Bagemihl[7] states that E is countable. Let

$N = E \setminus J F_s$.

St s

N is countable. Let $<p_Q$ denote the restriction of $<p$ to C — N.

If U is any open set, write $U = S_n$, where S_n £ S, S_n CZ U. Suppose x £ $(pQ^1$ (U). Then $(x) = <p(x)$ £ S_n for some n, which implies that a; £ acc $(/^{\sim 1}$ $(S_n C \setminus D))$. Thus

¥>0 W) C $\setminus J$ acc (f- $\setminus S_n$ n Z>)) - N. ***n***

On the other hand, suppose x £ acc IP D)) for some n, and x 4 N. Choose an arc y in $fXSn$ $(P$ $D)$ with one endpoint at x. Clearly,

C(f, $y)$ C S_n $(P$ D $CS_n CU$.

Since x 4 E,

?.$()$ = $p()$ £ $C(j,$ $y)$ c u,

so x t ^'(G). Thus

$\setminus J$ acc $(T^l(S_n$ $r \setminus D))$-NC ^$\setminus U)$, n

SO

$<Po'(U)$ = U acc $(r \setminus S_{,,} f^{\sim} \setminus$ $D))$ — N = $\setminus J$ $(F_s$, U $G_{s,,})$ - N ***n n***

= VGs. -n = (U^s.) n (C - N). n n

Thus, for each open set 17, $<Po^l(U)$ is an open set relative to $C^{\sim}N$. Therefore $<p_0$ is continuous. *Q.E.D.*

3. Boundary functions for continuous functions.

Definition. Let 8 and T be metric spaces. We will say the function / is of *Baire class* 1 (8, $T)$ *ij*, and only if,

(i) domain $j = 8$,

(ii) range IQT, and

(iii) there exists a sequence {/„} of continuous functions, each mapping 8 into T, such that $f_{,,}$—>f pointwise on 8.

We will say the function g is of *honorary Baire class 2(8, T)* if, and only if, (i) domain $g = 8$, (ii) range g C T, and

(111) there exists a function / of Baire class 1(8, $T)$ and a countable set N such that f Is-jv = Is-at .

Lemma 3. Let f be a continuous real-valued junction in D and let <pbe a finitevalued boundary junction jor j. Let r and t be real numbers with r < t. Then

(A) *there exists a G_s set G and a countable set N such that*

^([r, 4-oo)) o G 2 ^([t, +oo)) - AT, *and*

(B) *there exists a G_s set H and a countable set M such that*

$t])$ 2 H 2 <[1]((-°°, r]) - M.

Prooj. Let

[7] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat, Acad, Sci. U. S. A.*> 4 (1955) 379-382.

t — r e ~ 2 ’

C_n = Lit^1 | |z| = 1 - 4 , (n/

$\boldsymbol{A_n}$ = \ $\boldsymbol{ztR^2}$ | 1 - - < >| < 4 ,

(n J

E_n — {x e C | there exists an arc y at x having one endpoint on C_n , with y — {x} Q f^1((— ^{00}, r))|,

K = {x e C | there exists an arc y at x with y — {$} c f^1^ — $_c$, +«>))}.

Observe that

^((-co.r)) C \ jE_n , n«l

and

<\(/ - e, +«>)) C K.

For the time being, let n be a fixed integer. If x £ K, we can find an arc y_9 at x such that

y* - M C A_nC\ f\ \ t - $_c$, +«>)).

Since an arc at x is by definition a simple arc, y_x — {x} is a connected set. It follows that y_x — {rc} must be contained entirely within one component of the open set

$A_n$nn(/-e,+-)).

We denote this component by U_x . U_x is a nonempty open connected set._ Let T be the set of all points of K which are two-sided limit points of E_n.

Assertion. If x, y £ T and x 1 y, then U_x C\ U_y = <j>.

To prove this assertion we assume that z is a point of U_x A U_v and we derive a contradiction. Choose points x^r and y’ in y_x — {x} and y_v — {?/} respectively. Join x to x^f by an appropriate subarc of y_x . Join x^f to z by an arc in U_x . Join z to y* by an arc in U_v. Join y’ to y by a subarc of y_v. Putting these arcs together, we obtain an arc a with endpoints at x and y such that

a - {x, y} C A_nC\ f~\(t - e, + co)).

We can assume that a is a simple arc, for if a is not a simple arc we can replace a by a simple arc a’ CZ a having endpoints at x and y (see[8]). a is a crosscut of D. Let Li and L_2 be the two open arcs of C determined by x and y. According to[9] (Theorem 11.8, p. 119), D — a has two components, Pi and V_2 , whose boundaries are L_r kJ a and L_2 a respectively. From the fact that C_n is connected and does not intersect a it follows that C_n is contained entirely within one component of D — a. By symmetry, we may assume C_n £ V 2 •

Since $ is a two-sided limit point of E_n , Lr must contain a point of E_n , and hence a point of E_n . Say w £ A E_n . There exists a simple arc ft joining w to some point on C_n , with

ft — {w} C ^{00} , r)).

ft — {w} cannot have a point in common with a, because

[8] H. Tietze, Uber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.,* 5 (1919), 284-291.

[9] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

a ~ $\{x, y\}$ C f~$\backslash(t$ - ϵ, $+^{00}))$, and

e,+-)) =</> .

Thus C_n U (0 — {w}) is a connected set not meeting a. C_n U (/3 — {w}) meets y_2,soC_nVJ (0 — {w}) C V_2 . Consequently, w is in the boundary of V_2 . But this is a contradiction, because w e L_x and the boundary of V_2 is L_2 U a. This proves the assertion.

From the assertion it follows immediately that T is countable; for any family of disjoint nonempty open sets is countable. We know that the set >S of all points of E_n which are not two-sided limit points of E_n is countable.

K $C\backslash$ E_n = [K $C\backslash$ S] U [KH (E_n - S)] = ($KC\backslash$ 8) T.

This shows that (for any ri) K $C\backslash$ E_n is countable. So if we let

N = K $C\backslash$ $\backslash JE_n$ =. Q (K $C\backslash$ E_n), n = l n=l

then N is a countable set. Let

G = C — _)E_n . n = l

G is a G_s set. Using the fact that

<*((-«>, r)) C $\backslash jE_n$ C $\backslash jE_n$, **n=l n—1**

we find that

00

C - <*((- oo, r)) 2 c - $\backslash jE_n$ = G 2 K - N.

n=l

But

=^([r,+-))

and

K O ^((/ - ϵ, +«>)) o +«>)), so

<X[r, + +oo)) - N.

This proves (A). To prove (B), simply replace / and by —f and and apply (A).

Theorem 2. Let f be a continuous real-valued function in D, and let <p be a finite-valued boundary function for f. Then is of honorary Baire class 2(C, R).

Proof. For each pair of rational numbers r and t with $r < t$, choose G& sets $G(r, t)$, $H(r, t)$ and countable sets $N(r$, i), $M(r, f)$ such that

+«>)) O $G(r, t)$ O + <»)) - $N(r, f)$, and

<*((- ~, t]) 2 $H(r, f)$ 2 «>, r]) - $M(r$, Z).

Let

N = {J [$N(r, t)$ V $M(r$, /)],

where the union is taken over all pairs of rationals r, t with $r < t$. N is countable. Let (p_0 denote the restriction of to C — N, and let G*(r, t) = $G(r, t)$ — N. Since every countable set is an F_{ff} set, G*(r, t) is a G_8 set. Observe that

(2) ^([r, +-)) = ^([r, +«.))- N 2 G*(r, f)

2 +-)) - N = ^([Z, +-))•

If t is a fixed rational number, let $\{r_n\}$ be a strictly increasing sequence of rational numbers converging to t. Then, by (2),

$C\backslash$ <PoX[r. , +»)) 2 O G*(r. , f) 2 <p~$_0\backslash$[t, + «>)) = ^$\backslash$[r_a ,+«)), n=l n=l n=l

SO

^'(R,+°°)) = $C\backslash G^*(r_n,f)$. n = l

This proves that for every rational *t,* ^([J, + ^{00})) is a G_8 set.

If u is any real number, choose a strictly increasing sequence {£nJ of rational numbers converging to u. Then

+^{00})) = fWoWn , +^{00})), n = l

SO <p-Q\[u, + oo)) is a G_8 set. By a similar argument, we find that ^o^1((— ^{00}, *u])* is a Ga set for every real u. So

>, + -)) = (C - *N) n (C - , u]y)*

is the intersection of an F_{ff} set with C — N. By a theorem stated on p. 309 of Hausdorff's paper[10], <p0 can be extended to a real-valued function on C such that for every real *u,* +^{00})) is a G_8 set and +<»)) is an F_a set. By Theorem IX of the same paper, is of Baire class 1(C, 7?). Since *<p(x) = pxtx)* except for *xz N, <p* is of honorary Baire class 2(C, 7?). *Q.E.D.*

Corollary. Let f be a continuous -junction mapping D into R^N, and suppose <p : C —» R^N is a boundary junction jor j. Then <p is of honorary Baire class 2(C, *R^N).*

Proof. We simply write our functions in terms of their components, say

f — (fl ! *f'2 J * ' •* , and *(p — (epi , <p2 i • • • , <Pn) '*

Obviously *<Pi* is a boundary function for /,• , and so is of honorary Baire class 2(C, jR). We choose a function of Baire class 1(C, *R)* which agrees with *<pi* except on a countable set *M i* . Setting

0 = <0i , 02 , • • • , 0AT>,

it is clear that g is of Baire class 1(C, *R^N),* and that g agrees with *<p* except on the countable set VJi-i *Mt* • Hence *<p* is of honorary Baire class 2(C, *R^N).*

Q.E.D.

Lemma 4. Let g be a continuous junction mapping C into R^3. Let q be a point of R^3 and let e be a positive real number. Then there exists a continuous function g^* : C —> 7?^{3} *such that q does not lie in the range of g^*, and for all x v* C,

\g(x) - q e => *g(x) = g*(x).*

Proof. Let

S = *{yvR^3* / *\y* - g| < e}.

If 0(C) *CZ S,* let g^* : C —> R^3 be any continuous function whose range does not include q. Otherwise, 0^{-1}(/S) is a proper open subset of C and hence can be written in the form

g~XS) = UA, ***k***

where

Ik = {e'' | *at < t < b_k],* and

k 1 *I* =>I_k *Ii* =</> .

[10] F. Hausdorff, Uber halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.,* 5 (1919) 292-309.

Since $0^{\sim 1}(\{0\})$ is a closed (and therefore compact) subset of $0^{\sim 1}(>S)$, $0^{\sim 1}(\{\hat{}\})$ is covered by a finite number of I_{fc}'s. Say

0'1({0}) ... \ JI_n.

The endpoints e^{iak} and e^{t6A} of I_k are not in $0^{\sim 1}(\{<?\})$, so we can construct, for each *k,* a continuous function $g_k : I_k$-+ R^3 such that

$Sk(e^{iai}) = g(e^{iai}), = g\hat{}*),$

and q is not in the range of g_k . Define

0^*(x) = 0(a:), if o;$_£$C~ (AUZ$_2$U ... U/$_n$),

0^*(x) = $g_k(x)$, if x e I_k , k = 1, • • • , n.

It is easy to show that g* has the desired properties.

Theorem 3. Let f be a continuous function mapping D into the Riemann sphere 2, *and let <p be a boundary function for f. Then <p is of honorary Baire class* 2(C, 2).

Proof, Since 2 is a subset of J?^{3}, the corollary to Theorem 2 shows that $<p$ is of honorary Baire class 2((7, 2?^{3}). Let g be a function of Baire class 1(C, 2?^{3}) which differs from $<p$ only on a countable set N, Then $g(C)$ — 2 is countable, so there exists a point q inside of 2 (that is, in the bounded open domain determined by 2) which is not in the range of g. Let {</„} be a sequence of continuous functions converging to g. By Lemma 4 we can find (for each ri) a continuous function $g^*_n : C$ —» R^3 such that q does not lie in the range of g^* , and for all x £ C,

- g| $g_n(x) = g^*(x)$. *a*

It is easy to show that g^*_n —» g.

We define a function P as follows. If a e R^3 — $\{q\}$, let I be the unique ray with endpoint at q that passes through a, and let $P(a)$ be the intersection point of I with 2. Obviously, P is a continuous mapping of R^3 — {<?} onto 2, and P fixes every point of 2. Therefore

F(0(z)) = if $x\hat{}N$,

$P(g^*_n(x))$ is a continuous function from C into 2, and

F0(x)) as n oo.

This shows that $<p$ is of honorary Baire class 2(C, 2). *Q.E.D.*

4. Boundary functions for Baire functions.

In this section we concern ourselves only with real-valued functions. We shall prove that a boundary function for a function of Baire class a 1 is of Baire class a + 1. It is convenient to prove this theorem for functions that are defined in the (open) upper halfplane and have boundary functions defined on the rr-axis rather than for functions defined in D, Once the theorem is proved in this form it is a routine computational matter to show that it also holds for functions defined in D, The reader should be familiar with the results of Hausdorff[11] before reading this section. Unfortunately, we must begin with some tedious preliminaries.

[11] F. Hausdorff, Uber halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.,* 5 (1919) 292-

Let

We will regard C° as being identical with *R.*

Suppose *8* is a metric space. Let g^ be the class of all open sets of $ and let be the class of all closed sets of *8.*

A function / : >8 —> *R* is of *Baire class* 0 if and only if it is continuous. For any ordinal number *a* > 0, *f* is of *Baire class a* if and only if / is the pointwise limit of a sequence of functions each of Baire class less than *a.*

Let denote the class of all sets M C S such that

M = r\(r, +«>)),

for some real rand some function / of Baire class *a* on >S. Let 912 denote the class of all sets *N* C & such that

N = r\[r, +-)),

for some real r and some function / of Baire class *a* on *8.* It is easily shown that 9TCj = and 9l£ = .

Let

9 — 9c«> = g«,

gj = ,

9E^a = ,

91* = 9lco = 912 , If 0 is any class of sets, let 0_a denote the class of all countable unions of members of 0, and let 0_8 denote the class of all countable intersections of members of 0. Each of the following facts is either explicitly stated in[12], or can be easily deduced from statements found in[13], or is obtained by a routine transfinite induction argument.

I. 9TC2 = *(\J* 9lX , 9i? = (U 9TtX •

X<a X<a

II. Let *A* be any subset of the metric space 5. If / is a function of Baire class *a* on *S,* then / 1^ is a function of Baire class *a* on *A.*

III. Let *f* be a function of Baire class *a* whose domain contains {*(x, b)* / *x* e 2?}. Then *j((x, b))* is a function (of *x)* of Baire class *a.*

IV. If *A* C *8,* then

9R? = *{M r\ A* / *M* e Oil?}, 9Z" = *{N C\ A* | Ve9l?}.

V. If / is of Baire class *a* on 8, then for each real *r,*

and

VI. If a 2, then $(g_s)_s$ U C gn" *C* .

VII. *E £ °fC^a$_s$o 8 - E* e .

VIII. ^{c}JTCs and are closed under finite unions and intersections. is closed under countable unions and 91^ is closed under countable intersections.

309.

[12] F. Hausdorff, Uber halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.,* 5 (1919) 292-309.

[13] F. Hausdorff, Uber halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.,* 5 (1919) 292-309.

IX. Let f be a real-valued function on S. Suppose that for every real r

and

r>, +«))e^ .

Then / is of Baire class *a.*

Definition. If A and B are two sets, we will call A and B *equivalent,* and write A ~ B, if and only if A — B and B — A are both countable. It is easily verified that ~ is an equivalence relation.

Lemma 5. If A ~ E, then 8 — A ~ 8 — E for any set 8. If A_n ~ E_n (for all n in some countable set V), *then*

)jA_n ~\j$E_n$ and ~ •

ntN ntN ntN ntN

The proof of this lemma is routine.

Definition. An interval of real numbers will be called *nondegenerate* if it contains more than one point.

Lemma 6. Any union of nondegenerate intervals is equivalent to an open set.

Proof. Let 4 be a family of nondegenerate intervals and let H = For any x and y let

ZO, *y)* = *[x, y},* if x g y,

and let

i(%>y) = *[y,* ^1,

Define a relation (R on H by

x(Sly <=>I(x, y) CZ H,

(x, yzH).

It is easy to show that (R is an equivalence relation on H. In view of the fact that a set A of real numbers is an interval if and only if

x, y e *A =>I(x, y) C A,*

it is obvious that each equivalence class is an interval. For each *x £ H,* there exists an *I £ $* with *x £ I.* Every member of I is equivalent to x. Thus each equivalence class contains more than one point, and hence is a nondegenerate interval. Let $\{J_a\}$be the family of equivalence classes. Any disjoint family of nondegenerate intervals is countable, so there are only countably many J_a's. Let E be the set of all endpoints of the various J_a's. Then E is countable and

H = \J J_a ~ IJ J_a - E = \J*j*,* ***a a a***

where <7* is the interior of J_a . This proves the lemma.

Lemma 7. Let h be an increasing real-valued -function on a nonempty set E CZ R, Suppose that \|x — *h(x)*\| 1 *for every x* s *E. Then h can be extended*

to an increasing real-valued function h_x on R.

Proof. Let *e* ~ g.l.b. E (e may be — oo). For each x_Q e (e, + °°) set

hi(x<f) = l.u.b. *{h(x) / x* e (— oo , x_Q\| *C*\| *E}.*

Since *Jx* — *h(x)*\| g 1 for all x e E,

x e (— oo, x_0] *TA E h(x)* g x_0 + 1, so is finite-valued. If e = — oo we are done. If e > — oo} then *x e E* implies

h(x) e — 1, so A is bounded below. For x_Q s (— °°, *e]* set

Ai(^o) = g.l.b. {A(x) | *xtE],*

It is easily verified that h_r has the desired properties.

Lemma 8. Let f be a real-valued function of Baire class a on R. Let h be an increasing real-valued function on R. Set g(x) = Then there exists a countable set N such that g tv *is of Baire class a.*

Proof. It is well known that an increasing function has at most countably many discontinuities. Let *M* be the set of discontinuity points of *h.* If *f* is of Baire class 0, then *g* is continuous at all points of I? — *M,* so *g* is of Baire class 0. This proves the lemma for the case *a* = 0.

We now proceed by transfinite induction. Suppose the lemma holds for every ordinal X < *a.* If *f* is of Baire class *a* we may choose a sequence of functions *{f_n}* converging to *f,* where f_n is of Baire class X_n < *a.* If we set g_n*(x)* — f_n*(h(%y)* it is clear that $g_{}!}$*(x)* —» = {/W- By the induction hypothesis we may choose (for each n) a countable set N_n such that g_n It?-^ is of Baire class X_n . Let *N* = *Nn* . Then g_n $/_{R\_N}$ is of Baire class X_n , and since g_n |t?_at —» *g* U-at, *g* / -at is of Baire class *a.* This proves the lemma.

Theorem 4. Let f be a real-valued function of Baire class a 1 *on D°} and let <p be a finite-valued boundary function for f. Then <p is of Baire class a* + 1.

Proof. Let *r* and *t* be two real numbers with *r* < *t. r* and *t* will remain fixed throughout the first part of the proof. Set

Q = +<»)),

E = P\J Q,

t — *r* e = – •

Observe that *P C\ Q* = *<t>.* For each *xtE,* choose an arc y_x at *x* such that lim f*(z)* = *<p(x),* y_x *C {z /* \z — x| g 1}, ***z—>x***

and

(a) /(?*) £ (— ^{00} , r + e), if *x* s *P*

(b) f($_7$») £ (J - $_\epsilon$, +«>), if *xtQ.*

(This is accomplished by cutting the arc off sufficiently close to a;.) We remark that if *x e P* and *y £ Q,* then y_x *C* y_y =</> .

We will say that y_x meets y_y in $A^Q{}_n$ provided that y_x and y_y have subarcs y' and y' respectively such that *x* £ y' *GZ A°, y* £ y' *C A°,* and y' P y_y 4</> . Let

L_o = *{x £ P* | (Vn)(3?/ 4= *x)(*y_x meets y_y in *A°*$_n$*)},*

L_r = *{x £ Q* I (Vn)(3?/ 4= *x)(*y_x meets y_y in A®)},

M_o = *£ P / (ln)(*y_x meets no y_v *(y* 4= *x)* in A®)},

{x £ Q / (?ri)(y_x meets no y_y *(y* 4= *x)* in A®)},

L = Lo V Lt ,

M = M_0*\J* M_r.

Observe that L_o , L_r , M_o , *Mt* are pairwise disjoint, and that *P* = L_o M_o and *Q* = L_x U M_r .

For each *x £ M,* let n_x be an integer such that y_x meets no y_y (with *y* 4= *x)* in A®* . Notice that *n* n_x implies y_x meets no y_y in A® . Let

$K_n = \{x$ £ E / y_x meets C®, and if x £ M, n_x $n\}$.

Clearly E = VJn-i K_n . Moreover, K_n C K_{n+1} for each n.

Take any fixed integer n. For each x £ L_o we can find a y 4= x such that y_x meets y_v in A® . Let $I^n{}_x$ be the nondegenerate closed interval between x and y. We shall show that $I^n{}_x$ C L_o U (C® - K_n\ If t £ $I^n{}_x$, either t £ C^Q - K_n or t £ K_n . Suppose t £ K_n . Then y_t meets C®, and (if t £ M) n_t n. It is clear from Figure 1 that y_t must meet either y_x or y_y in A® . (This can be rigorized by means of Theorem 11.8 on p. 119 in[14].)

Consequently, t % M. Now x iL_0 Q P, so since y_x intersects y_y , y $ Q. So y e E — Q = P. Similarly, since y_t meets y_x or y_y, t e E — Q = P. Thus t e P — M = Lq . We have shown that t e $I^n{}_x$ implies that t e C° — K_n or t e L_q , so $I^n{}_x$ £ Lq V (C° — K_n). It follows that (for each ri)

$L_0Q(\backslash Jr\hat{}r\backslash E$ $q[l_0\backslash j$ $(c^\circ$ - xj] n e. $XzLq$

Let $W_„$ = $kJxet$, $I^n{}_x$. By Lemma 6, W_n is equivalent to an open set. $l_oq(c\backslash w\}c\backslash e$ 'n==l /

C M $[L_o$ kJ $(C^\circ$ - K$_n$)]} H E = $\backslash l_0$ v $f\}$ $(.0^\circ$ - K„)} n E

= $\{L_o$ n $E]$ H (C° - K„) n $e\backslash$ = L_o V = L_a .

Therefore L_Q = $W_n)$ A E. Since each W_n is equivalent to an open set there exists a G_Q e g$_8$ such that

L$_o$ — Go n E.

Similar reasoning shows there exists a G$_x$ e $Q_§$ such that

E.

Next we study the properties of M_Q , It is convenient to define a function 7T : R^2 —» R by $ir(\{x, y))$ = If x e M $C\backslash$ K_n , then, starting at x and proceeding along y_x , let $(?_n(x)$ be the first point of $C^Q{}_n$ reached. Set $h\hat{}(x)$ = $ir(a_n(x))$ (for x e M $C\backslash$ Kn).

is an increasing function on M $C\backslash$ K_n ; for if Xi, x_2 £ M K_n and x_x < x_2, then, since cannot meet y_{Xi} in $A^Q{}_n$, it is evident (see Figure 2) that $tt\hat{}Xxj))$ < ^(<r$_w$(£$_2$)). (The argument can be rigorized by means of Theorem 11.8 on p. 119 in[15].) Since

y_x £ $\{z$ | |z - 1}, |x - $h^\circ{}_n(x)\backslash$ g 1.

So by Lemma 7 can be extended to an increasing function h_n on C°.

Let

Gn(x) = •

For x e M $C\backslash$ K_n ,

gn(x) =

If x £ M, then for all sufficiently large n, x £ M $C\backslash$ K_n , so lim $g_n(x)$ = lim f(<r„(a;)) = $<p(x)$. ***n—*a>*** n—⊠<»

Thus g_n k —» cp k . By III, /(($, 1/n)) is a function (of x) of Baire class a, so by Lemma 8 we can choose, for each n, a countable set N_n such that g_n / $_C$*-N_n is of Baire

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[14] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

[15] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

class a. Let N = VJZi N_n . Then g_n k-# is of Baire class a. But g_n Iat-jv —> $<p$ k-w , so $<p$ k-^ is of Baire class $a + 1$.

Now

p (M - N) = *&* k-k^{-1}((- r]) = *Try (m- n),* where T s 9l^{a+1} (by IV and V). Clearly *P C\ M* ~ *T C\ M.*

We have

L = P_o U *L,* ~ (G_o n F) VJ (G_t n *E)* = (Go U *GJ H E,*

so L ~ *G C\ E* where G e g_8 . Also

M_0=PKM~TC}M = T C\(E - L)

~*tc\[e* - (g n P)] = *[t n (c° -* g)] n *e.*

Since G e g_5 , G° - G e , so by VI and VIII, *T C\ (C°* - G) e 9T^{+1}. Thus

M_o — T_o n *E,*

where T_o e 9l“$^{+1}$. Now we can examine the properties of *P.*

P = (G_qKE)U (T_o n *E)* = (Go U To) n *E,*

so, again by VI and VIII,

p ~ 7\ n p,

where T_x e 9l^{a+1}. Since a countable set is in and the complement of a countable set is in g_8 , it is easy to show (using VI and VIII) that

P = T_2 H P,

where T_2 e 9l^{a+1}. Since *P C\ Q = <f>}*

P £ T_2 C C° - *Q.*

Remembering the definitions of *P* and *Q}* and observing the fact that C° — ^$^{-1}$(k +^{00})) = ^$^{-1}$((” 00, 0)> we can summarize the results of the first part of the proof as follows.

For each pair r, t of real numbers with r < t, there exists a set T(r, t) e 3l“$^{+1}$ *such that*

^((~ r]) C *T(r, t)* C a,, *t».*

Given any real r, let {Z_n} be a strictly decreasing sequence of real numbers converging to *r.* Then

<’((—^{00}, »•]) = °°, /„)). n = l

So

<*((- »,r]) £ H *T(r, /„)* C *f* <^{1}((- «>, i_n)) = v~\(- «, r]), n“l *n*—1

and hence

^((_ro,r]) = C\T{r,Q. n~l

By VIII,

<^{1}((~°o,r]) e9T^{+1}.

Since / is an arbitrary function of Baire class a in Z)° and $<p$ is an arbitrary boundary function for /, we can replace /, $>, r by — /, — *cp,* ~r to find that

^([r, +«>)) e9l^{a+1}.

Also,

^((r, +00)) = *C°* - £snr^{+1}.

By IX, $<p$ is of Baire class $a + 1$. *Q.E.D.*

5. Boundary functions for measurable functions.

Theorem 5. Let f be a real-valued Borel-measurable function in D^Q and let <p be a finite-valued boundary function for f. Then <p is Borel-measurable.

Since every Borel-measurable function is of some Baire class a, this theorem is an immediate consequence of Theorem 4. We now show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable.

Let *u* denote Lebesgue measure on *R* and let /z^2 denote Lebesgue measure on R^2. Let denote exterior Lebesgue measure on *R]* that is,

Me(S) = g.l.b. {/z(G) I *G* is open and *E* C *G},*

for any set *E C R.*

Lemma 9. Let h be an increasing real-valued function on a set E C R. Then there exists an open interval I 2 *E such that h can be extended to an increasing real-valued function on I.*

Proof, If *E* is unbounded below, set a — — co. If *E* is bounded below, set *a* = g.l.b. *E,* if (g.l.b. *E) 4 E,*

a = (g.l.b. *E)* - 1, if (g.l.b. *E) £ E.*

If *E* is unbounded above, set *b* ~ + 00. If *E* is bounded above, set

b = l.u.b. *E,* if (l.u.b. *E) 4 E,*

b = (l.u.b. *E)* + 1, if (l.u.b. *E)* e *E.*

Let *I* = *(a,* b). Clearly *E c:* Z. Let *e* = g.l.b. *E (e* may be — <»). For x_0 £ (e, *b)* set

ftxtf) ~ l.u.b. *{h(x) / x* c (a, $\$_0$] Pi *E}.*

If *e* = *a* we are done. If *e* >*a* then *e £ E.* For x_Q £ (a, *e]* set f(x_0) = A(e). It is easily verified that *f* is finite-valued and increasing, and is an extension of *h.*

Lemma 10. Let E C R be a set of measure 0 *and let h be an increasing function on E. Suppose h(E) has measure* 0. *Then {x + h(x) / x £ E} has measure* 0.

Proof. Extend *h* to an increasing function *g* on an open interval *I* = *(a,* b) 2 *E.* Set *g(a)* = — 00 and *g(b)* = + 00. Take any e > 0. Choose an open set *G* such that *I* and *utG)* < e/2. Choose an open set *H Z) h(E)* with /z(ZZ) < e/2.

Say

G = U I_n , and *H* = *\J* J_m , *mN mtM*

where *{I_n / n £ N}* and *{J_m / ms M}* are countable families of disjoint open intervals. Let I_n = *(a_n* , b_w), and observe that a_n , b_n £ [a, b]. Set

s = *(J {g(a„), g(b_n)}* - {- 00 , + co }.

mN

Notice that *S* is countable. Set

K_n = (g(a„), g(&n)).

One can easily verify that *k* 1 *n* implies K_k P K_n = 0.

If *A* and *B* are two subsets of 2?, let

A + *B* = {a + b \ *as, A,b sB}.*

It is easy to show that for any two intervals J and <7', g_e(<7 +</') g + ^(J'). Let W = *{x + h(x) / x £ E].*

Assertion.

w c *(E + S) KJ \J \J* [(Z_n n *g-W)* 4- *(J_n r\ K.)]. mN imM*

To prove this, let w be an arbitrary point of *W.* Write $w = x + h(x)_f$ where *x sE.* For some n, x £ I_n . Since g is increasing,

h(x) = g(x) £ [#«), g(b_n)].

If *h(x)* equals $g(a_n)$ or #(6_n), then *h(x)* e *S,* so $w = x + h(x)$ e E + 5. On the other hand, suppose h(x) 4= #(a_n), <?(&„). Then *h(x) £ K_n* . Also, *g(x) = h(x) £ J_m* for some m. Thus *h(x) £ $J_m C\ K_n$* and *x £ I_n C\ g~\J_m)}* so that

w = *x + h(x)* £ (I_n *C\ g~\J_m)) + (J_m C\ K_n).*

This proves the Assertion.

Since g is increasing, *g^J^* is an interval, so both *I_n C\ g~\J_m)* and *J_m C\ K_n* are intervals. Also note that m 4= *I* implies *g~\J„^ g~\Ji) —*</> . By the Assertion,

M.W g *+ s)* + S E n *g~\$j_m$y)* + ($J_w$ n #$_n$)] ***mN mtM***

g *n_e(E + s)* + E E [m(/$_b$ n n o

mN mzM

= m/U (s + t?)) + E [E n *g~\$j_m$y) + £ »($j_m$ c\ K_n)]* ***szS mN mtM mtM***

< E + E) + E *W.)* + E mGA. n 2Q]

utS mN mtM

= 0 + ju(G) + E E *c* ***K_n) nzN mzM***

= m(g) + E E m(A. n *K_n)*

mtM nzN

m(G) + E mW = m(<?)+ ***1EH)*** < e. ***mt M***

Since e is arbitrary, *jn_e(W)* = 0.

Lemma 11. Let L — {(x, a) / x £ E} and M = {{x_y b) / x £ R} be two horizontal lines in R^2. Let E be a set of (linear) measure 0 *in L and let F be a set of (linear) measure* 0 *in M. Let £>be a set of closed line segments such that*

(a) , $2 £ <£, *S2* ☒—Si *F\ S2* =

(b) $£ <£=>*one endpoint of s lies in E and the other endpoint lies in F.*

Let S = \$J_{kt\mathcal{E}}$ s. Then u^2(S) = 0.

Proof. Assume without loss of generality that *b >a.* For any *(x, y) £ R^2* let *^({x, y)) = x.* For any *y £ R* let *l_y = {{x, y) / x £ R}.* Let

E_o = {z £ E / z is the endpoint of some s £ <£},

and observe that *E_o* has linear measure 0. For any set *A* C *R^2* we of course set tt(A) == *{x £ R / (xj y) £ A* for some *y £ R}.*

We define a function *h* on *ir(E_0)* as follows. If $ £ *^(Eq),* then *{x, a} zE_0* , so we can choose a (unique) segment s £ <£ with one endpoint at (x, *a).* If the other endpoint of s is p, we set *h(x)* = 7r(p). Clearly *h* maps *ir(E_0)* into *tt(F).*

Since the segments in £ cannot intersect each other, *h* must be an increasing function.

Take any y_Q with $b > y_Q > a$. Let $c = b — y_0$, $d = y_0 — a$. A simple computation shows that if q s l_{V_o} $C\backslash$ 8, then

tt(q) =

ex + $dh(x)$ $c + d$

for some x s tt(£?$_0$)- So

7r(l_{yo} A 8) C

ex + dh{x) . $e + d$

X £ 7r(Fo)

Now $(d/e)h(x)$ is an increasing function mapping $tt(E_q)$ into (d/c)7r(F), so by Lemma 10

$x + h(x)$

X £ $Tt(Eq)$

has measure 0. Hence

X £ $Tt(Eq)$

ex + dh(x) . $e + d$

X £ $Tt(E_0)$

has measure 0, so A $SY) = 0$. But A 8)) = 0 also when y_Q 4 (a, b), so A 8)) = 0 for every y. If we knew that 8 were measurable, the lemma would follow immediately from the Fubini theorems. But since we have, as yet, no guaranty of the measurability of 8, a more complicated argument is necessary. At several stages in the argument the reader will find it useful to draw diagrams to help him visualize the situation.

For any y_Y , y_2 s R, let

$U(,yi >$ 2/2) = $\{\{x, y) \mid x, y\ t\ R, y_t < y < y_2\}$.

A set of the form $U(yi , y_2)$ will be referred to as a *horizontal open strip.*

For each positive integer n, let <£(n) denote the set of all segments s e £ such that s has a point in common with $\{\{Xj\ b) / x$ e $(—n, ri)\}$. Let

$S(n)$ = [U $_S$]H + i b-i).

Since l_a and l_b have (plane) measure 0, and since

sc I.U4U Q $S(n)$, $\boldsymbol{n = l}$

it is sufficient to show that each $S(ri)$ has measure 0.

Let n be a fixed positive integer. Set a* = a + 1/n and b* = b — $1/n$. Take any e > 0. Choose €$_0$ so that 2e$_0$ + e^* < e/(b — a). Let y_0 be any member of [a*, b*]. For the time being, y_Q will be held fixed.

For each s e <£, let p_8 be the endpoint of 8 on l_b , let q_t be the intersection point of s with l_{V9} , and let r$_a$ be the endpoint of 8 on l_a .

Choose an open set G £ R such that $tt(1_{V_q}$ A $S(n))$ C G and g((?) < e$_0$ • Say G = VJ, Ij, where Z, = (a$_f$, b^ and the Z, 's are pairwise disjoint. We may assume that each Z,« contains a point of $tt(1_{V9}\ S(n))$. For each let

Ci = *g.l.b.* {7r(pa) | s e £(n), 7r(ga) e ZJ,

dj = *l.u.b.* {^(p,) | s e £(n), 7r(g«) e Z,}, c< = *g.l.b.* {?r(ra) | 8 e <£(n), ?r(^) e Z,}, *d'i* = *l.u.b.* {?r(ra) | 8 s £(n), *ir(q8)* e ZJ.

Note that c, g d,- and c< g d< . Since the segments in £ cannot intersect each other, it is easily seen that the intervals (c,- , d,) are all pairwise disjoint. It is also clear (from the definition of £(n)) that each (c,-, d,) is a subset of (~ n, ri). Hence, if we set a,- = d,« *c7-* , we have 22,- a,- g *2n.*

For each *j,* let *s(j)* be the line segment joining the two points (c< , *a),* , b), and let *t(j)* be the line segment joining the two points (d< , a), (d,- , *b).* Let A_f be the closed subset of *U(a,* b) which is enclosed by the two line segments 80), ^0). Let *Hj* denote the intersection of A,- with the horizontal open strip

V — LI max

€o I • *J 7* I €0 I I o'^-^pminV'^ + 2^/r

Note that H_f is measurable. Setting ZZ = H_f, *it* is clear from the definition of the A/s that

S(ri) C\ V QH.

Take any *y* s *R.* We wish to show that

M0-(ZZ A ly)) <

__ e__ *b* — *a*

We can, of course, assume that

l J **€o I • J 7 . €o I |**

y e I max *\a,* y_a — , mm S *b,* y_0 + ^(r

An elementary computation, using the geometrical properties of *Hj,* shows that

n i,)) g (i + *4*^-

\ *o* y_Q / *o* y_Q

Therefore

n z„)) g E n *w*

- G+n ^)e°+2n2

" 2c0 + €0 < t , *b* — *a*

so *v(ir(H* l_v*Y)* < *e/(b* — *a)* for every *y.*

We have shown that *for each* y_Q e [a* &*] *there exists a horizontal open strip V(*y_Q*) containing* l_{V_0} *, and there exists a measurable set H(*y_Q*) C* y(?/0), *such that S(n) H V(*y_0*) £ H(*y_0*)*

and (for every y) ir(H(y_Q*)* l_v*) is measurable and*

y«H(y_0*) Cy* Zy)) <

The various open strips VG/o) (i/0 « [a*, 6*]) clearly cover the compact set {(0, *y}* I *y* e fa> *&*]} • Choose a finite subcovering *V(*y_2*\ • • • , V(*y_m*).* Set

tm— 1 /

m \

U *V(y,)]*

J"»+l /

n I7(a*, 6*).

$H(y_m)$ U \J IW) -

X

Obviously K is measurable, and for each y, $ir(K$ A Z,,) is measurable and $C\backslash\ l_v)) < e/(b - a)$. Moreover, $S(n)$ C K. We have

\K) = £ ^(K C\ *1,))* dy *g* £ dy = *(6* - a*) < *e.*

Since e is arbitrary, this shows that

g.l.b. {/(X) | K measurable, $S(n)$ C $K\}$ = 0.

Therefore $S(n)$ has measure 0.

Lemma 12, For every e > 0 there exists a strictly increasing function h on R such that h(R) has measure 0, *and for every x,* I# — A(x)| e

Proof. For each (not necessarily positive) integer n, let I_n = [ne, (n + l)e]. Then I_n = R. There exists a strictly increasing function f : [0, 1] [0, 1]

such that m(/([0, 1])) = 0. For example, such a function may be defined as follows. Any number in [0, 1) may be written in the form

$.a_1a_2a_3 \bullet\bullet\bullet a_n \bullet\bullet\bullet$, (binary decimal),

where the decimal does not end in an infinite unbroken string of l's. Set

$f(.a!a_2a_3 \bullet\bullet\bullet a_n \bullet\bullet\bullet) = bib_2b_3 \bullet\bullet\bullet b_n \bullet\bullet\bullet$, (ternary decimal), where bi = 0 if = 0 and 6,- = 2 if a,- = 1. Set f(l) = 1. f maps [0, 1] into the Cantor set, so m(/([0, 1])) = 0. It is easily shown that / is strictly increasing.

For each n, it is easy to obtain from / a function f_n :I_n-> I_n such that j_n is strictly increasing and /z(fn(In)) = 0. Set

$h(x)$ = f_n(x) for x e (ne, (n + l)e].

There is no difficulty in proving that h has the desired properties.

Theorem 6. Let be an arbitrary junction on C^Q = *{{x,* 0) | x e $R\}$. Then *there exists a junction j on D° — {{x, y) / y >* 0} *such that j(z)* = 0 *almost everywhere and <p is a boundary junction jor j.*

Proof. For each positive integer n let h_n be a strictly increasing function on R such that $y(h_n(R))$ = 0, and for every x, $]x — h_n(x)j$ ~ *1/n.* Let and E_n has linear measure 0. For each n, x let $s_n(x)$ be the line segment joining $(h_n(x)$, 1/n) and (A_{n+1}(a;), l/(n + 1)). Since

E_n is a subset of

$h_n(x) > h_n(x') \ x > x^f => h_{n+1}(x) > h_{n+1}(x'\backslash$

we find that x 4= x^f implies $s_H(x)$ $C\backslash$ $s_n(x^f)$ = 0. Since each $s_n(x)$ has one endpoint in E_n and the other in E_{n+1} , Lemma 11 shows that for each n

= 0.

x&R

Hence

M^2(0 Us_n(a:)) = 0. 'n=^{s}l ***xtR*** •

Set

$j(z)$ = £>(($, 0)), if z e $s_n(x)$ for some n,

$f(z)$ = 0, if z is not in any $s_n(x)$.

$j(z)$ = 0 almost everywhere. Let

y(x) = {{x, 0>} U $0s_n$(x). 74 = 1

Since the endpoints of s_n($) are at ($A_n$(z), *1/n)* and *($h_{n+1}$(x)j l/(n* + 1)), and since *(h_n(x), 1/n)* —> (x, 0) as n —> oo, it is clear that *y(x)* is an arc at *(x}* 0). Obviously

lim *j(z)* = *<p({x_t* 0)).

This proves the theorem.

Corollary. There exists a measurable function in D° having a nonmeasurable boundary function.

6. Concluding remarks. Our theorem on boundary functions for continuous functions could have been proved by a small modification of the argument in Section 4, but the proof in Section 3 is shorter and neater.

The reader may wonder whether Theorem 4 holds true for functions taking values on the Riemann sphere as well as for real-valued functions. The theorem does, in fact, remain true in the sphere-valued case. If we regard the Riemann sphere 2 as a subset of R^3 and apply Theorem 4 to each component of *f* and *<p,* we find that *<p* is of Baire class *a* + 1 with R^3 as the universal range space. It is then easy to show by means of Satz 2 in Banach's paper[16] that *<p* is of Baire class *a* + 1 with 2 regarded as the universal range space. A similar procedure shows that Theorem 5 also remains true for functions taking values on the Riemann sphere.

The results of Sections 2, 3 and 4 cannot be extended to three dimensions—at least not in the most obvious way. We can show this as follows. Let *K* be an open cube in R^3 and let *F* be one face of *K.* If *f* is defined in A, then we say *<p* (defined on *F)* is a *boundary function* for *f* provided that for each *x* e *F* there exists an arc *y* with one endpoint at *x* such that *y* — {x} *C K* and

lim *f(v)* = *(p(x). v*—>Ⓡ *vzy*

Lemma 13. Suppose that every point of F is an ambiguous point of the function f : K —» R^3. Then f has a nonmeasurable boundary function.

Proof. Let *E* be a nonmeasurable subset of *F.* Since each point of *F* is an ambiguous point we can choose, for each *x* e *F,* two distinct points *^(x), <p_2(x)* e R^3 such that there exist arcs *y{* at *x* with

lim *f(p)* = *Pitx), (i* = 1,2). *v—*x vsy i*

Let

<p(x) = <pi(o:), if *x vE,*

cp(x) ~ if *x* e *F* — *E.*

Then

<p(x) — *^(x)* =0, if *xtE,*

<p(x) — *Vi(x)* 1 0, if *x s F* — *E.*

Therefore *(<p* — ^i)$^{-1}$({0}) = *E,* so — *<p_x* is not a measurable function. Hence either *<p* or *<p_x* is a nonmeasurable function. Since *<p* and *<p_Y* are both boundary functions for /, the lemma is proved.

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[16] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17 (1931) 283-295.

P. T. Church[17] has constructed an example of a homeomorphism / from *K* onto *K* such that every point of *F* is an ambiguous point for f. By Lemma 13, *f* has a nonmeasurable boundary function <*p.* Theorem 1 is therefore false in three dimensions. Write f and <*p* in terms of their components; say / = (f i, f_2, fa) and <*p* = , <?2, ^3). Since <*p* is nonmeasurable, one of its components, say <?,- ,

is nonmeasurable. But is a boundary function for the continuous real-valued function f,« , so Theorem 2 and Theorem 4 must be false in three dimensions.

References

The University of Michigan

[17] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

5. 1966 - On a Boundary Property of Continuous Functions

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Explanation by John D. Bullough

The author generalizes the result of McMillan (1966) to the effect that the set of curvilinear convergence of a continuous function f from D into Z is of type F(sd). The generalization considers f as a continuous function from D into a compact metric space E. Topologizing the set of closed sets C(E) of E with the Hausdorff metric and letting *E* be any closed set in C(E), it is shown that the set of all x (C such that there is a boundary path v at x with the cluster set of f along v contained in some set of *E* is of type F(sd). Taking *E* to be the set of all singletons {y}, y (E (which is closed in C(Z)) McMillan's theorem is obtained.

Various other corollaries are given by selecting appropriate closed sets *E* (C(E).

Article by Ted

On a Boundary Property of Continuous Functions

T. J. Kaczynski

Let D be the open unit disk in the plane, and let C be its boundary, the unit circle. If x is a point of C, then an *arc at* x is a simple arc y with one endpoint at x such that y - {x} c D. If f is a function defined in D and taking values in a metric space K, then the *set of curvilinear convergence* of f is

{x e C| there exists an arc y at x and there exists a point p e K such that lim f(z) = p} .

Z —> X zey

J. E. McMillan proved that if f is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of f is of type $F_a$$ [2, Theorem 5]. In this paper we shall provide a simpler proof of this theorem than McMillan's, and we shall give a generalization and point out some of its corollaries.

Notation. If S is a subset of a topological space, S denotes the closure and S* denotes the interior of S. Of course, when we speak of the interior of a subset of the unit circle, we mean the interior relative to the circle, not relative to the whole plane. Let K be a metric space with metric *p.* If x_0 e K and r > 0, then

S(r, x_0) = {x e K| p(x, x_0) < r} .

An arc of C will be called *nondegenerate* if and only if it contains more than one point.

LEMMA 1. *Let bea family of nondegenerate closed arcs of* C. *Then* Uie^ I " Uje/Z I* *countable.*

Proof. Since U_{Ie}^r I* is open, we can write I* = U_n J_n, where {J_n} is a countable family of disjoint open arcs of C. If

X_0 € U I - U I* , Ie# Ie#

then for some I_o e #, x_0 is an endpoint of I_o. For some n, Iq c J_n, so that x_0 e • But x_0 / J_n, so that x_0 is an endpoint of J_n. Thus Uj I - Uj I* is contained in the set of all endpoints of the various J_n; this proves the lemma. ⊠

In what follows we shall repeatedly use Theorem 11.8 on page 119 in[1] without making explicit reference to it. By a cross-cut we shall always mean a cross-cut of D. Suppose y is a cross-cut that does not pass through_the point 0. If V is the component of D - y that does not contain 0, let L(y) = V Cl C. Then L(y) is a nondegenerate closed arc of C.

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I

Suppose n is a domain contained in D - { 0}. Let r denote the family of all cross-cuts y with y n D c n. Let

1(0) = U L(y), I_0(n) = U L(y)* .

yE r yE r

Let ace (n) denote the set of all points on C that are accessible by arcs in n.

The following lemma is weaker than it could be, but there is no point in proving more than we need.

LEMMA 2. *The set* ace (n) - I0(n) *is countable.*

Proof. By Lemma 1, I(n) - I0(n) is countable; therefore it will suffice to show that ace (n) - I(n) is countable. If ace (n) has fewer than two points, we are done. Suppose, on the other hand, that ace (n) has two or more points. If a E ace (n), then there exists a' E ace (0) with a' =f. a. Let y, y' be arcs at a, a' , respectively, with

ynncn , y'n n cn.

Let p be the endpoint of y that lies in n, p^ the endpoint of y[1] that lies in n. Let y" c n be an arc joining p to p' . The union of y, y', and y[11] is an arc o joining a to a'.

[1] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17

By[2], there exists a simple arc 0[1] co that joins a to a^. Clearly, 0[1] is a cross-cut with 0[1] n D c n and a, a^ e L(0 '). Thus a e I(n), and so ace (n) c 1(0). •

LEMMA 3. *Suppose* 01 *and* Oz *are domains contained in* D - {O}. If

(1) IqCOj) A acc(O_1) *and* Iq(Q_2) A ace (Q_2)

are not disjoint, then n1 *and* O_z *are not disjoint.*

Proof. We assume n1 and Oz are disjoint, and we derive a contradiction. Let a be a point in both of the two sets (1). Let Yi be a cross-cut with Yi n D cni such that a E L(yi)* (i = 1, 2). Let Ui and Vi be the components of D - Yi, and (to be specific), let Ui be the component containing 0. Note that y_1n D and y_2n D are disjoint.

Suppose yi nD cVz and *yz A* D c Vj. Then, since yi n D c Ui, Ui has a point in common with Vz. But O E u_1 nUz, so that U 1 has a point in common with Uz also. Since U_i is connected, this implies that U_i has a common point with Yz n D, which contradicts the assumption that Yz nD cVp Therefore yi nD ^ V2 or Yz n D ^ Vi • We conclude that either y 1 nD c U_2 or y_2 nD c U1 . By symmetry, we may assume that Yz n D cU_i.

It is possible to choose a point b E L(yi) * that is accessible by an arc in Oz, because a is in the closure of ace (Oz). Let y be a simple arc joining b to a point of y_4 n D, such that y - { b} c Oz. Then y - { b} and y_i are disjoint. Also, y — i_b} contains a point of Ui (namely, the point where y meets y2 n D); therefore y - {b} c Ui . Hence b e Ui. Since b e L{y 1)*, this is a contradiction. •

THEOREM 1 (J. E. McMillan). *Let* K *be a complete separable metric space, and let* f *be a continuous function mapping* D *into* K. *Let*

X = {x E CI *there exists an arc* y *at* x *for which* lim f(z) *exists}* . z^x

zEy

Then X *is of type* F_a $.

Proof Let $\{p_k\}_{k=1}$ be a countable dense subset of K. Let {Q(n, be

a counting of all sets of the form

where *0* is a rational number. Let {U(n, m, k, £)}$^{\wedge}_{=1}$ be a counting (with repetitions allowed) of the components of

(We consider 0 to be a component of 0.) Let

A(n, m, k, £) = acc[U(n, m, k, £)].

Set

co co co co

$_Y$= n u u u I_0(U(n, m, k, £)) Cl A(n, m, k, £).

n=l m=l k=l f = 1

Since I_0(U(n, m, k, £)) is open, it is of type F_a . It follows that Y is of type $F_{a<}$5 .

I claim that Y c X. Take any y e Y. For each n, choose m[n], k[n], f [n] with

(2) y € I_0(U(n, m[n], k[n], f[n])) A A(n, m[n], k[n], £[n]) (n = 1, 2, 3, •••).

(1931) 283-295.

[2] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

For convenience, set U_n = U(n, m[n], k[n], f[n]). By (2) and Lemma 3, U_n and U_{n+}i have some point z_n in common. For each n, we can choose an arc y_n c U_{n+}i with one endpoint at z_n and the other at z_{n+1} . Then y_n c Q(n + 1, m[n + 1]). Also,

and therefore

r r

P<Pk [n] ' Pk [n+r] } P(Pk[n+i-l]' P_k[n+i]) < < ^2'

Thus {p_k[n]} is a Cauchy sequence and must converge to some point p g K. Because

r_n C U n+1 C f'1 (S(^iT>Pk[n+l])) and Pk[n]^P>

lim f(z) = p. It is possible that y is not a simple arc, but by[3] we can replace y z-> y zG y

by a simple arc y' c y. Thus y e X, and we have shown that Y c X.

Suppose x g X. Let y_0 be an arc at x such that f approaches a limit p [1] along

y_0. Take any n. Choose k with p' g

p_k j . Choose m so that x is in the

interior of Q(n, m) A C. Then y_0 has a subarc yjj, with one endpoint at x, such that

t'q - {x} c Q(n, m) n f-lfs/X p_k) j .

Hence, for some f, x g acc[U(n, m, k, £)] = A(n, m, k, f). This shows that

00 00 00 00

X c n u u u A(n, m, k, £).

n=l m=l k=l f = l

By Lemma 2, the set

A(n, m, k, J?) - I_0(U(n, m, k, $.)) = A(n, m, k, £) - [$l_0$(U(n, m, k, £)) Cl A(n, m, k, £)] is countable. It follows by a routine argument that

A U A(n, m, k, jO - A U [l_0(U(n, m, k, £)) A A(n, m, k, *I)]* n m,k,£ n m,k,£

is countable. Because

A U [l_Q(U(n, m, k, £)) A A(n, m, k, £)] = Y c X C A U[1] A(n, m, k, £), n m,k,j£ n m,k,f

the set X - Y is countable, and therefore X is of type F_a 5 . ☒

Before stating our generalization of the foregoing theorem, we must say a few words about spaces of closed sets. If K is a bounded metric space with metric p, let ^(K) denote the set of all nonempty closed subsets of K. Hausdorff [1, page 146] defined a metric *p* on MK) by setting

p(A, B) = max{ sup dist(a, B), sup dist(b, A)} ,

a GA b€B

where dist(x, E) denotes inf p(x, e). If K is compact, then $?(K) is a compact

e € E

metric space with *p* as metric [1, page 150].

If f maps D into K and if y is an arc at a point x e C, we let C(f, y) denote the cluster set of f along y; that is, we write

[3] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,*

C(f, y) = {p 6 K| there exists a sequence {z_n} c y AD such that z_n -» x and f (z_n) —> p} .

THEOREM 2. *Let* K *be a compact metric space, and let* 8 *be a closed subset of* ^(K). *Let f: D —>K be a continuous function. Then*

{x e c| *there exists an arc y at x and there exists*

E e 8 *such that* C(f, y) c E}

is a set of type F_a $.

Proof. If s > 0 and E e ^{r}^(K), let

Z/$^?$(8, E) = {a e k| there exists be E with p(a, b) < 8} .

Note that <^(8, E) is open and that

F e ^(K), p(E, F) < 8 =>Fc <^(8, E).

Let {P(k)}$k_{=1}$ be a countable dense subset of 8 (such a subset exists, because every compact metric space is separable). Let

X = {x e c| there exist an arc y at x and an E e 8

such that C(f, y) c E} .

r t ^{00}

Let {Q(n, m)}$_{m=}$j be defined as in the proof of the preceding theorem. Let {U(n, m, k, 4)}£$_=$i be a counting (with repetitions allowed) of the components of

^{p}<^{k}>)) ^{n} Q(^{n}> ^{m}>-

Let A(n, m, k, £) = acc[U(n, m, k, £)], and set

CO CO CO OO

$_Y$= n u u u I_0(U(n, m, k, £)) n A(n, m, k, f).

n = l m=l k = l £ = 1

Since I_0(U(n, m, k, £)) is open, it is of type F_Q . It follows that Y is of type F^ .

I claim that Y c X. Take any y e Y. For each n, choose m[n], k[n], f[n] so that

(3) y e I_0(U(n, m[n], k[n], £ [n])) n A(n, m[n], k[n], £[n]).

Set U_n = U(n, m[n], k[n], £ [n]). Since 8 is compact, there exist a P € 8 and some strictly ascending sequence {n;}°°$_{=i}$ of natural numbers such that

J J

P(k[$_{nj}$]) $_7$ P. ^{J} J

By (3) and Lemma 3, $U_{n>}$ and U_n. have some point Zj in common. For each j,' J i choose an arc yj c with one endpoint at Zj and the other at Zj_{+1}. Then

7j c Q(nj_{+1}, m[nj_{+1}]). Also,

y e A(n_{j+1}, m[n_{j+1}], k[n_{j+1}], £[n_{j+1}]) c U c Q(n_{j+1}, m[n_{j+1}]), *J*

and therefore each point of y. has distance less than ———— from y. Now

J Hj + l

2tt + 1 | |°°

0 as j —> therefore, if we set y = {y} U Ui=i , then y is an arc with ^{n}j+l

J J

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4 (1957) 155-156.

one endpoint at y.

I claim that C(f, y) c P. Take any p e C(f, y). There exists a sequence {w_s}^$_{=1}$ in y - {y} such that w_s y and f(w_s) p. Let e be an arbitrary positive number. Choose j_0 so that p(P(k[nj]), P) < e/3 for all j > j_0 . Choose so that j > jj implies l/nj_{+i} < e/3. We can choose an s such that w_s e *y*^ for some i > j_0 , jj and such that

(4) p(f(w), p) <-L

o

Then

f(w_s) e f(y.) c f($U_{n \cdot H}$) c P(k[n_{i+1}])) ,

and therefore we can choose a point q *e* P(k[n_{i+1}]) with

(5)

P(f(w), q) < — » n. ☒

E

3*

Moreover, because p(P(k[n_{i+1}]), P) < e/3, there exists some q' € P with

(6)

p(q, q') <

Together, (4), (5), and (6) show that p(p, q') < e. Since P is closed and e is arbitrary, this proves that p € P. Hence C(f, y) c P e 8. By[4], we can if necessary replace y by a simple arc y^1 c y; it follows that y 6 X. Thus Y c X.

Now suppose x e X. Choose an arc y_0 at x such that C(f, y_0) c P_o for some P_o e 8. Take any n. Choose k with p(P_0, P(k)) < 1/n. Then

$^{p}o^{c}$^(^ w) >

hence C(f, y_0) ^{c} P(k) j .

Choose m so that x is in the interior of Q(n, m) A C.

If for each natural number t there exists a point z'$_t$ e y_0 A S x j AD with $^{p(k)}$)) '^{then}

f(z') e K -*S* (i P(k)\,

and since K - *SP* (P(k) j is compact, there exist some a e K - *SP* P(k) j and a subsequence {f(z')}.°\ such that f(z') =-> a. But then a € C(f, y_0), contrary to ^{t}i ☒ \ ^{r}i

the relation C(f, y_0) ^{c} , P(k)) . We conclude that there exists a natural number t for which

y_0 n s(|,x) o C P(k))) •

It follows that y_0 has a subarc y^ with one endpoint at x such that

y'$_Q$- {x} c P(k))) nQ(n, m).

Hence there exists an £ such that

x e acc [U(n, m, k, £)] = A(n, m, k, £).

This shows that co oo oo co xcn u u uA(n, m, k, £). n=l m = l k=l £ = 1

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4 P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

By Lemma 2, the set

A(n, m, k, £) - I_0(U(n, m, k, £)) = A(n, m, k, J?) - [l_0(U(n, m, k, £)) A A(n, m, k, {)] is countable. It follows easily that

A U A(n, m, k, £) - A U [l_0(U(n, m, k, £)) A A(n, m, k, £)] n m,k,£ n m,k,£

is countable. Since

A U [I_0(U(n, m, k, £)) A A(n, m, k, 4)] = Y c X c A U A(n, m, k, £), n m,k,£ n m,k,£

X - Y must be countable. Thus X is the union of an $F_a$$ -set and a countable set, , and hence it is of type F_a $. ⊠

In each of the following four corollaries, let f denote a continuous function mapping D into the Riemann sphere.

COROLLARY 1 (J. E. McMillan). *Let* E *be a closed subset of the Riemann sphere. Then the set*

{x e C | *there exist an arc y at* x *and a point* p e E *such that* lim f(z) = p} z —>x z € *y*

is of type $F_{a\,6}$.

COROLLARY 2. *Suppose* d > 0. *Then the set*

{x *e* C| *there exists an arc y at x such that* [diameter C(f, y)] < d}

is of type $F_{Q\,6}$.

COROLLARY 3. *Let* E *be a closed subset of the Riemann sphere. Then the set*

{x e C | *there exists an arc y at x with* C(f, *y)* c e}

is of type $F_{a\,6}$.

COROLLARY 4. *The set*

{x e C| *there exists an arc y at* x *such that* C(f, y) *is an arc of a great circle }*

is of type $F_{Q\,6}$.

We can obtain all these corollaries by taking 8 to be a suitable family of closed sets and applying Theorem 2. To prove Corollary 4, we need the fact that C(f, y) is always connected. One could go on listing such corollaries ad infinitum, but we refrain.

It is interesting to note that in Corollary 1 it is not necessary to assume that E is closed. By combining Corollary 1 with Theorem 6 of[5], one can prove that the conclusion of Corollary 1 holds even if E is merely assumed to be of type .

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J., 8 (1961) 201-207.

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BOUNDARY-FUNCTIONS
by
Theodore John Kaczynski

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan

1967

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BOUNDARY FUNCTIONS

By Theodore John Kacijnski

Abstract

Let H denote the set of all points in the Euclidean plane having positive y-coordinate, and let X denote the x-axis. If p is a point of X, then by an arc at p we mean a simple arc v, having one endpoint at p, such that v - {p} (H. Let f be a function mapping H into the Riemann sphere. By a boundary function for f we mean a function t defined on a set E (X such that for each p (E there exists an arc v at p for which

lim f(z) = t(p).
z -> p
z (v

The set of curvilinear convergence of f is the largest set on which a boundary function for f can be defined; in other words, it is the set of all points p (X such that there exists an arc at p along which f approaches a limit. A theorem of J.E. McMillan states that if f is a continuous function mapping H into the Riemann sphere, then the set of curvilinear convergence of F is of type F(sd). In the first of two chapters of this dissertation we give a more direct proof of this result than McMillan's, and we prove, conversely, that if A is a set of type F(sd) in X, then there exists a bounded continuous complex-valued function in H having A as its set of curvilinear convergence. Next, we prove that a boundary function for a continuous function can always be made into a function of Baire class 1 by changing its values on a countable set of points. Conversely, we show that if t is a function mapping a set E (X into the Riemann sphere, and if t can be made into a function of Baire class 1 by changing its values on a countable set, then there exists a continuous function in H having t as a boundary function. (This is a slight generalization of a theorem of Bagemihl and Piranian.) In the second chapter we prove that a boundary function for a function of Baire class e > 1 in H is of Baire class at most e + 1. It follows from this that a boundary function for a Borel-measurable function is always Borel-measurable, but we show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable. The dissertation concludes with a list of problems remaining to be solved.

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LIST.OF ILLUSTRATIONS

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INTRODUCTION

1. Preliminary Remarks

Let H denote the upper half-plane, and let X denote its frontier, the x-axis. If x£X, then by an arc at x we mean a simple / ———————————————

arc y with one endpoint at x such that y - Suppose that f

is a function mapping H into some metric space Y. If E is any subset

I of X, we will say that a function <p:E ->Y is a *boundary function* for f if, and only if, for each x€E there exists an arc Y at x such that

lim f(z) = ip (x) . z x z € y

The study of boundary functions in this degree of generality was initiated by Bagemihl and Piranran[1]. A function defined in H may have more than one boundary function defined on a given set E S X, but it follows from a famous theorem of Bagemihl[2] that any two such boundary functions differ on at most a countable set of points.

If f is defined in H, then the *set of curvilinear convergence* of f is the set of all points x£ X such that there exists some arc alj x along which f approaches a limit.. Evidently, this is the largest set on which a boundary function for f can be defined.

J. E. McMillan [10] discovered that the set of curvilinear convergence of a continuous function is always of type F and in this paper we show that every set of type in X is the set of curvilinear convergence of some continuous function. Next, we show that if is a function defined on a subset E of X, then <p is a boundary function for some continuous function if and only if *if* can be made into a function of the first Baire class by changing its values on at most a countable set of points. (This solves a problem of Bagemihl and Piranian [2, Problem ,1].) We then consider functions that are not assumed to be continuous, and we prove that a boundary function for a function of Baire class 5 1 is of Baire class at most 5+1 (thus proving another conjecture of Bagemihl and Piranian[3]). It follows from this that a boundary function for a Borel-measurable function is always Borel-measurable, and in the last section we show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable.

Most of the results appearing here have already been published ([8] and [9]). At the time I published these papers I did not expect to have to make use of this material for a dissertation.

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[1] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

[2] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat, Acad, Sci. U. S. A.>* 4 (1955) 379-382.

[3] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

2. Notation

R will denote the field of real numbers, n

R will denote n-dimensional Euclidean space.

Points in R^n will be written in the form {x.,..., x \ rather than (Xp..„ x_n) (to avoid confusion with open intervals of real numbers in the case n = 2).

If v€R^n, then |v| denotes the length of the vector v. 2*3'2

S denotes {vgR : |v| = 1}. S^z will be referred to as the *Riemann sphere.*

. Let

H = ' { <x,y>6 R^4 : y > 0} ? 1

H_n = { <x,y>£R^Z : £ > y > 0} X = { <x,0>: x€R} < ^{X}n= {<^{x}4> : x€R} •

We consider X as being identical with R. Thus, for example, <x,0>< <x' ,0^ means x £ x', and for p, q € X, the notations [p,q], [p,q), etc. refer to the obvious intervals on X.

If E is a subset of a topological space, then E" denotes the *i i*

it closure of E, E denotes the interior of E, and E' denotes the

* complement of E. Of course, if E is a subset of X, then E means the interior of E relative to X, not relative to the whole plane. In Section 7, we often denote two line segments by s and s'. Since the prime notation is never used for complementation in that section, there is no danger of confusing s' with the complement of s.

2

If f is a function defined in a subset of R , then f(x,y) means f(<x,y^). Thus we write f(z) for z£R and f(x,y) for x,y£R interchangeably. 1 1

3. Baire Functions

In this section we review the main facts concerning Borel sets and Baire functions, and we prove some results that will be needed later.

If C is any family of sets, let C_g be the family of all sets that can be written as a countable intersection of members of C, and let be the family of all sets that can be written as a countable union of members of C.

Suppose M is a metrizable topological space. Let p\m) be the family of all open subsets of M and let q\m) be the family of all closed subsets of M. If ? is an ordinal number greater than 1, let

P^5(M) = (^Jq^(M)) n<C

Q^C(M) = (UP^nOT)$_s$ n<€ 9

For any C, E € $Q^?$(M) <=>E' 6 $P^?$(M) .

For any subset L of M. E€ P^(L) (respectively Q^(L)) if and ***1 F F***

only if there exists a set D€ P (M) (respectively Q (M)) such that e = dal.

P^(M) and Q^(M) are closed under finite unions and finite intersections. P^(M) is closed under countable unions and Q^(M) is closed under countable intersections.

If n<5, then P^n(M) U Q^n(M) $P^?$(M)A $Q^?$(M).

Let F$(M) be the class of all F^ sets of M, and let Gg(M) be the class of all G$ sets of M.

2 2

P (M) = F (M) and Q (M) ' = G$(M).

Let Y be a metric space. For any family C of subsets of M we will say that a function f : M -> Y is of *class* (C) if and only if f"\u)€C for every open set U — Y.

The following definition of the Baire classes is somewhat different from the classical definition, but it seems more.convenient for our purposes. A function f : M -> Y is said to be of *Baire class* 0(M, Y) if and only if it is continuous. If 5 is an ordinal number greater than or equal to 1, then f is said to be of *Baire class* • 00

£J(M, Y) if and only if there exists a sequence of functions mapping M into Y, f being of Baire class n_n(M,Y) for some n_n < £, such that f ⊠* f pointwise.

If f : M + Y is of Baire class £(M, Y) and if L is a subset of M, then $f|_T$ is o'f Baire class 5(L, Y).

If K is a metric space, if g : K •* M is continuous, and if ⊠ f : M -> Y is of Baire class ?(M, Y), then the composite function f og is of Baire class €(K, Y).

If Y is separable and if f : M -> Y is of Baire class £(M, Y), then f is of class $(P^{\wedge+}$\m)) [4, page 294].

If Y is separable and arcwise connected, if g 1, and if f : M-> Y is of class $(P^{\wedge+}$\m)), then f is of Baire class ?(M, Y)[5].

For any 5, if f : M -> R is of class $(P^{\wedge+}$\m)), then f is of Baire class £(M, R)[6].

If l€$Q^{\wedge+}$\m) and f:L ⊠* R is of Baire class 5(L, R), then f can be extended to a function ? : M + R of Baire class 5 CM, R)[7].

We say that a function f : M •* R is *Borel measurable* if, and only if, for every open set u£r, f'^CU) is a member of the o-ring of subsets of M generated by the open sets.

If f : M ⊠> R is of some Baire class $?C^M$, R), then f is Borel- measurable, and, conversely, if f : M -> R is Borel-measurable, then f is of Baire class £i(M, R) for some countable ordinal number 5 [7, page 294].

The proofs of Lemmas 1 through 6 are based on standard techniques in the study of Baire functions.

Lemma 1. Let M be a metric space, and let E and F be two F^ sets in M. Then there exist two disjoint F$ sets A and B £ M such that

E-FGA and F - E *Q* B.

00 oo

Proof. Let E = E and F = I J F , where E and F are closed, n n n n

n=l n=l

[5] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

[6] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

[7] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

Then

B_n, F_n £ f_o(m)a g_6(M).

It is easy to check that F^fM) AG^fM) is an algebra (i.e., is closed under complementation, finite unions, and finite intersections). We inductively define a sequence of pairs of sets (A_n, B_n) as follows. Let

$A_x = E_x$, $B_x = F_x$ **A** A' .

For n > 1, let

n-1 n

A = E **n** $C\|$ B! , B = F **nfl** A! . n n^n ' n n" j

By induction, A_r, B_n € F\$(M) A G\$(M). Let **, CO 00**

A =**U** A_n , B =**U** B .

n=l n=l

Then A and B are F sets. Notice that

(j **B** **- £** **F** **O** A. **g** e ,

j=l 3 j=l 3

from which it follows that n-l ,

A = E a **(** **U** B.) **O** E **A** F' , n n z v — n *

1=1 J

and n ,

Bn ⊠ y 2 PnAB' •

3=1.

Therefore

00

A **5 LJ CE$_n$A** F') = E - F n=l

and

CO

B **“ U** CF$_n$AE') = F - E. n=l

It only remains to show that A **A** B = <j>. Suppose x 6 A **A** B. Choose £, m with x £ A and x € B . If £ > m, then £ > 1, so that

J6 in

Hence x € B' —a contradiction. On the other hand, if *St,* < m, then m —

m

f.nA A! *a* a; , 1=1 J

so that x€£A} —another contradiction. We conclude that AAB =

If E is a subset of a space M, we let x_E denote the characteristic function of E.

Lemma 2. Let L be a. subset of a metric space M, and suppose that

co

E G F^CL) /A GgfL). Then there exists a sequence $\{f_n\}_{n\_}$^-°f continuous real-valued functions on M such that f^ -> x_E pointwise on L.

Proof. Both E and L - E are in F^CL), so there exist sets E^, F^ £ F^fM) such that

E = E^a L and . L - E = F^n L.

By Lemma 1, there exist A, B € F$(M) such that A AB = <|>and
E^ - F^ G a, F^ - E^ G B. We have
E = AAL and . L - E = BAL.
00,00..
Write A =LR. b-U. ®n* where A^, are closed and A_n ^n+i' n=l n=l
B^ B for each n. By Urysohn's Lemma there exists a continuous function f_n : M -»☒ [0,1] such that
f fx) = 1 when x £ A
$n^{1, J}$ • n
f f x) = 0 when x £ B .
n n
CO
$\{f_n\}_{n=1}$ is the desired sequence.®

Lemma 3. Let L be a subset of a metric space M, f : L-> R a function of class (F_a(L)) that takes only finitely many different values.
Then there exists a sequence {f } .of continuous real-valued n n n=l
functions on M such that f_n -> f pointwise on L.

Proof. From Banach's Hilfssatz 3[8], we see that there exist real numbers a_n,..., a and sets
1 ;q .
Ep..., E_q€F_o(L)r»G_6(L)
such that
. q
f = E a. Xn •
. . i E.
J=1 J J
☒ **1 00**
If we choose for each j a sequence {f_n J J_n_^ of continuous real-valued.
functions on M such that f -> x_c pointwise on L, and if we set n _ n c j
☒ **00**
then is the desired sequence.®

Lemma- 4. Let L be a metric space, f a bounded real-valued function
• . co
on L of Baire class 1(L, R). Then there exists a sequence tf_n^n_j of real-valued functions on L converging uniformly to f, such that each f is of class (F^fL)) and takes only finitely many different values. '

Proof, f is of class (F^CL)) and the range of f is totally bounded, so an obvious modification of the proof of Banach's Hilfssatz 4[9] gives the desired result.®

[8] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

[9] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

Lemma 5. Let M be a metric space, L a subset of M, f : L + Ra /

' ⊠ co

function of Baire class 1(L, R). Then there exists a sequence

of continuous real-valued functions on M such that f + f pointwise on L.

Proof. We first* prove the lemma under the assumption that f is bounded. For any bounded real-valued function h, let •

Uh II = sup { |h(x) | : x£ domain of h} .

By Lemma 4 we can choose, for each n, a function g_{ri} : L + Rof class (F^CL)) such that g_n takes only finitely many different values and

Let

^{h}l «1 ⊠ ^{h}n $^=$ - «n-l ^{for} " >K

Then, for ri > 1,

UM ⁼ - ^{f + f} - «n-lll

i it it *

Each h n

values,

is of class (F_o(L)) and takes only finitely many different

* *1 00

so by Lemma 3 we can choose (for eachn) a sequence {$h_n{}^J$}^_^

of continuous functions on M such that h^ h_n pointwise on L.

Set

k^fx) = - || h_n|| if $h_n{}^J$'(x) <_ - || hjl

^{k}n^j« = H^{h}nH $^{i£\ h}$$_n$^{j}W> H^{h}nll

^{k}n^j $^{W\ =\ h}$n$^{j\ Cx)\ i£}$ " II ^{h}nll $^{<\ h}$n$^{j\ (x)\ <}$ H ^{h}nH *

Then is continuous, k^ j- h_n pointwise on L, and || k^ ||' £ II ^$_n$ll < —. Therefore, if we set 2$^{n“2}$ CO

f. = Z k j ,

J i n

^{J} n=l

then the series converges uniformly and fj is continuous on M. We claim that ff pointwise on L. Take any x £ L and any e > 0. Choose m large enough so that —< 4 ^{e*} For each ^{n}> choose j (n) so 2^m that

j - M*)l <^1-

Let i = max {j (1) ,..., j (m) }. Then j >_ i_Q implies that

|f (x) - f(x) | < |f.(x) - E k ^{j}(x)| + | E $^{3\ 3}$ n=l ^{n} n=l

m

$_+$ I z n=l

“ -ii ^{m}

< Z II $k_n{}^3$ II + E |$k_n{}^3$(x) -h_n(x)l + || g_m n=m+l n=l

n m , $_n$

< —-iy + (E —i=-)|+ —= e.

< $_o$m-2 ^{k} . „n+l ' 3 _m 3

z n=l 2 2
Thus f. fx) ⊠* f(x) for each x € L, and the lemma
m k \x) - E h (x)| n=l
h_n« - £(X)|
- f II
is proved for bounded J J f. __ _
If f is not bounded, let
g(x) = arctan f(x)
Then - < gW < y for every x € L, and g is of Baire class 1(L, R),
00
so there exists a sequence {g_n^ n _j °f continuous functions on M converging to g pointwise on L. Set
h_nM = -14 if . g_n(xj < - 41
h_n(x) =4-1 if g_n(x)> 1-1
h_n(x) = . ^(x) if . ' . 1< ^(x) < ' . 1 .
71 7T
Then h_n is continuous on M, - < h_n(x) < and Ir ⊠* g pointwise on
L. Let f (x) = tan h (x). Then f is continuous on M and f -> f n n n n
pointwise on L.l '
Lemma 6. If L is a subset of a metric space M and f : L ⊠> R^m is a function, then the following are equivalent.
(i) f- is of Baire class 1(L, R^m) .
(ii) f is of class (F^fL)). **00**
(iii) There exists a sequence <^ n ^ n—j of continuous functions mapping M into R^m such that f •* f pointwise on L.
This lemma is an easy consequence of Lemma 5.
3
Definition. Let q be any point of R lying inside the bounded open 2 3 2
domain determined by S . By the q-*projection* of R - {q} onto S we
mean the function P^ defined as follows. If a is any point of 3
R - {q}, let *I* be the unique ray, having its endpoint at q, that
passes through a, and let P^(a) be the intersection point of *I* with 2 3 2
S . Pq is a continuous mapping of R {q} onto S that fixes every 2
point of S .
Theorem 1. Let L be an arbitrary subset of R . Then a function **2** 2
f : L S is of Baire class 1(L, S) if and only if it is of class (F_a(L)).
Proof. Assume that f : L -> S is of class (F_o(L)). S $s R , so by
Lemma 6 there exists a sequence of continuous functions mapping
2 3
R into R such that f -> f pointwise on L. Let
A n </sup> V^{1 C {v € r3 :} l^V> <sup> 7}
11 11 "

^{B}n = V^{1} C {v€ r3 : M ' S 7})
11 11
' C_n = V^{1} ({vg : H
11 11 *L**

Let f ° = f_n|^ • According to [5, Lemma 2.9, page 299], f^^{0} can be n 2 3 1
extended to a continuous function g_n : R -> {v €. R : |v| = y}. 2 3
Define h^ : R -> R - {0} by setting
h_nCx) = g_n(x) if x 6 B_n
h_n(x) = f_n(x) if x € C_n.

Since B_n, are closed, h_n is continuous, and it is easy to verify that h_n(x] ⊠* f(x) for each xG L. Let k_n : R^ •* be the composite
function P o h . Then k is continuous, and for each x£L, o n n
2
k_n(x) -> P_o(f(x)) = f(x). Thus f is of Baire class 1(L, S

Definition. Let M and Y be metric spaces. Then a function f : M Y
is said to be of honorary Baire class 2(M, Y) if and only if there exists a countable set N^m and a function g : M ⊠> Y of Baire class
1(M, Y) such that f(x) = g(x) for every x €. M - N.
2

Theorem 2. Let L be an arbitrary subset of R and let Y be either
2
the real line, a finite-dimensional Euclidean space, or S . Then a function f : L •* Y is of honorary Baire class 2(L, Y) if and only if there exists a countable set N C L such that f|is of class (F_q(L-N)).

Proof. Suppose that f : L •* Y is of honorary Baire class 2(L, Y). Then there exists g : L ⊠> Y of Baire class 1(L, Y) and a countable set N£L such that $f|_{L N}$ = gl_{L_N}* But $gI£_{\_N}$ is of class (F_o(L - N)). Conversely, suppose that f|^ is of class (F_o(L - N)), where N is countable. We must show that f is of honorary Baire class 2(L, Y). First consider the case where Y = R^m. Write
f(x) = Xf^x), f_2(x), ...,
Then f^|^ is of class (F^CL - N)) (i=l,..., m), and it follows that fflb jq is of Baire class 1(L - N,*R). Since L - N €G_g(L), we can extend f. $|_T$ to a function g. : L -> R of Baire class 1(L, R) . If we set g(x) = <^g^(x),..., g_m(x)} , then g is of Baire class 1(L, R^m) and g(x) = fCx) for x € L - N, so we have the desired result.
2 2 3
Now consider the case where Y = S . Since SCR, there 3 exists, as we have just shown, a function g : L ⊠* R of Baire class 3 2
1(L, R) such that g(x) .= f(x) for all x € L - N. Then g(L) - S is countable, so there exists some point q in the bounded open domain
determined by_S such that q(£ g(L) . Let h be the
2
P *o* g. Then h maps L into S , and for each x £ L 4

h(x)
P.(gM) = P (f(x)) 4 4
composite function “ N, f(x).
2
If (JC S is open, then

$h^{-1}(U)$
$^{-1}$(P ”^U)) € F CL) ,

so h is of class (F (L)) . By Theorem 1, h is of Baire class
2 M
1(L, S), so we have the desired result.Ⓡ

CHAPTER I. BOUNDARY FUNCTIONS FOR CONTINUOUS FUNCTIONS

If r is a positive number and.if y_Q is a point of a metric space Y having metric p, then

S(r, Y_o) denotes ’ {y £ Y : p(y, *yj* < ri.

We will repeatedly make use of Theorem 11.8 on page 119 in [11] without making explicit reference to it. This theorem states that if D is a Jordan domain in R or in R U {“}, if y is the frontier of D, and if a is a cross-cut in D whose endpoints divide y into arcs y^ and y£, then D-a has two components, and the frontiers of these components are respectively a u y_1 and *a* u (The term cross-cut is defined on page 118 in [11].)

4. Domain of the Boundary Function

Definition. If f is a function mapping into a metric space Y, then ‘ the *set of curvilinear convergence* of f is defined to be

{x € X : there exists an arc y at x and there exists y € Y such that lim f(z) = y}.
z ->x zg y

J. E. McMillan [10] proved that for suitable spaces Y, the set of curvilinear convergence of a continuous function is always of type Ftffi . We- Ⓡive a more direct proof of this result than McMillan’s.

(This proof can be modified to give a more general result; see [9].)

. 15

i i t I I 16 ⊠

* An interval of X will be called *riondegenerate* if and only if !
it contains more than one point. ⊠

Suppose y is a cross-cut of H. If V is the bounded component of H - y, let L(y) = V f|X. Then L(y) = [c, d], where c and d are the

• - *J* endpoints of *y* and c < d. Suppose Q is a domain contained in H. Let r denote the family of all cross-cuts *y* of H for which *y* H C fl, and let

1(8) = U L(y)*. Y^r

Let acc(fl) denote the set of all points on X that are accessible by arcs in fl.

Lemma 7. Assum^ that acc (fl) is nonempty. Let a be the infimum of acc (fl) and let b be the supremum of acc (fl). Then

I (fl) = (a, b) .

Proof. Suppose x€.I(fl). Let y be a cross-cut of H such that *'it **

Y *fs* H £ fl and x € L(y) • L(y) = [c, d], where c and d are the endpoints of y and c < d. It is evident that c and d are in acc(fl), so a £ c < x < d £ b, and x € (a, b). Conversely, suppose x* (a., b) • Then there exist points c', d' € acc(fl) with c' < x' < d'. Since fl is arcwise connected, it is easy to show that there exists a crosscut y' of H, with y'^HSa, having c', d' as its endpoints. But then x' € (c', d') = L(y') , so x' € I(fl).☒

Lemma 8. If fl., and fl_ are domains contained in H, and if X M

(1) I (fl) A acc(flj) and I(fl_2) H acc(fl_2)

are.not disjoint, then and Q_2 are.not disjoint.

Proof. We assume that and fi_2 are disjoint and derive a contradic

tion. Let a be a point in both of the two sets (1). Let y^ be a ☒ cross-cut of H, with A such that a g L(y^) (i = 1, 2). Let

IL and AL be the components of H - y^, where is the bounded component. Observe that y^ A H and y_2 /'Mi are disjoint.

Suppose yj H H C v_2 and y_2 AH G V^. Then, since y^AH $Up has a point in common with V_2« But, since U.^ is unbounded, U.^ cannot be contained in V_2, so must have a point in common with y_2 nH. This contradicts the assumption that y_2 A H £Vp so we conclude that either y^ A H^V_2 or y_2 AH^Vj. Hence, either y^ A H U_2 or y_2 A H By symmetry, we may assume that

y_2 HH$$U_r$

fi_2 does not meet y^, and fi_2 does meet U^ (because y_2 A H S U-i A ft?) > so fi_2 C U^. Since a € acc(fi_2), there exists a point b € L(yp such that b € acc(ft_2). But then b € fi_2£Up and this is * impossible because the frontier of U^ is disjoint from L(y^) . ☒ Theorem 3 (J. E. McMillan). Let Y be a complete separable metric space and let f : H -> Y be a continuous function. Then the set of curvilinear convergence of f is of type F* .

Proof. Let $\{p_v\}_v$"-. be a countable dense subset'of Y. Let {Q(n, m)} " K m=l

be a counting of all sets of the form

' { <x, y>: 0 < y < and r < t < r + i-}

where r is a rational number. Let'{U(n, m, k, £)} " be a counting 1

(with repetitions allowed) of the components of

i •

p_k))n Q(n, m) .

(We consider | to be a component of <j>.) Let
A(n, m, k, &) = acc[U(n, m, k, £)] .
Set
CO CO CO • CO
»• n u u u I(U(n, m, k, $,)) n A(n, m, k, £).
n=l m=l k=l £=1
Since I(U(n, m, k, &)) is open in X it is of type F . It follows that
O'
B is of type F . Let C denote the set of curvilinear convergence o 6
of f. I claim that B $C. Take any b <B. For each n, choose m[n],
k[n], £[n] with I
(2) b € I(U(n, m[n], k[n], a[n])) A A(n, m[n], k[n], £[n])
(n = 1, 2, 3,...).
For convenience, set U_n = U(n, m[n], k[n], £[n]). By (2) and Lemma 8,
U and U . have some point z in common. For each n, we can choose n n+1 n
an arc y_T i - ^{U}n+1 with one endpoint at z_n and the other at $^z{}_{n+}$-^« Then y_nCQ(n+l, m[n+l]). Also,
b € A(n+1, m[n+l], k[n+l], £[n+l]) S=U_{n+1} Q(n+1, m[n+l]),
2 and therefore each point of y_n has distance less than —from b.
2 l"l
—->0 as n ->«; hence, if we set y =' {b} U $y_{n>}$ then y'is an arc n=l
with one endpoint at b.
Since U and U , have a point in common, n n+1 r '
-11 -11
£ (St J? Pk[n]» and £ (S<JST- Pk[n_+1]»
have a common point, and hence ,
SC3P pk[np and SC3irT' pk[n+i:P z z
have a common point. Therefore, if p is the metric on Y, then
t pCpk[n]' Pk[n+lp - $_9$n + $_9$n+r < $_9$n-l *
and therefore
f _ y 1 1
pCpk[n]' Pk[n+rp - p(pk[n+i-l]' Pk[n+ip < $_2$n+i-2 x '
Thus {p_k[-$_n$p is a Cauchy sequence and must' converge to some point p € Y. Since
v_nSVlSf'1(s(^r> Pk{ntl]" "d
pk[n] ;p-
lim f(z) = p. It is possible that y is not a simple arc, but Z-*b
z€Y according to [12] we can replace y by a simple arc
y' C y. Thus b € C, and.we have shown that B^C.
Suppose c € C. Let y_Q be an arc at c such that f approaches a limit p' along y_Q. Take any n. Choose-k with p' € P_k) • Choose m so that c is in the interior of Q[h, m) A X. Then y_Q has a subarc y_Q', with one endpoint at c, such that
Y_o' - {c} S Q($_n$, m) n f"1CS(~, p_kB .

Hence, for some £, c £ acc[U(n, m, k, £)] = A(n, m, k, s.) . This
shows that
CO CO 00 co
csQUUUA(n, m, k, £) n=l* m£l k=l «=1
It is easy to deduce from Lemma 7 that the set
A(n, m, k, £) - I(U(n, m, k, £)) = ...
A(n, m, k, £) - [I(U(n, m, k, £))nA(n_5 m, k, £)]
contains at most two points. It follows by a routine argument that
A(n, m, k, £) - k-J [I(U(n, m, k, £)) n A(n, m, k, £)] n m,k,£ n m}k,£
is countable. Since
[I(U(n, m, k, A)) A A(n,' in," k, £')] = B^c n m,k,£
n m,k,£
C - B is countable, and therefore C is of type F g. **B**

Next we will show that the foregoing theorem is as strong as possible, in this sense: if A is any set of type F $ contained in X, then there exists a bounded continuous complex-valued function f defined in H such that A is the set of curvilinear convergence of f. The proof is unfortunately quite long. .

Definition. Let E^ and E_2 be two sets on the real line. A point p on the real line will be called a *splitting point* for E^ and *E?* if either

Xj £ p for all x_x £ E^ and p £ *x? for* all x_2 G E_2 or x_2 £ p for all x_2 € E_2 p £ x^ f°r a*1 xi € E^.

We will say that two sets E^ and E_2 *split*, or that E^ *splits with* E_2, if and only if there exists a splitting point for E^ and E_2.

• co
Lemma 9. Let E be an set in R. Then there is a sequence $\{E_n\}_{n\_}$^
of sets such that
(i) E is bounded and closed n
(ii) if n m, then either E_n and E^ are disjoint or E_n and
E_m split
(ifi)
Proof. We can
n, and .
CO write E = I *J* A where n=l
A is closed, A CA. for all n n — n+1
Observe that if I is any open interval, then there exists a
CO
countable family {J } .of bounded closed intervals such that n n=l **00**
n m J and J split, and I = J^. Since any open set of
n=l
real numbers is a countable disjoint union of open intervals, it
00
follows that for any open U there exists a countable family H J

of bounded closed intervals
00
such that n i m I and T n
I split, and m r '
n=l
' r Ih 00
For each n, let {ipj_^ vals such that j k ZZ^ and
be a family of bounded **00**
I? split, and A = IJ j=l
closed inter
im. Let J
IF = {Ap U {l" n A_{n+1} : n = 1, 2,...; j = 1, 2,...}.
Then^7 is a countable family of bounded closed sets, and CO 00
E ⊠ - A 1^WU ($A_{n+1}AA_A$)
n=l n=l
= i") = A_{1U}C{
n=l]=1 J J J
⊠IK-
/ **J***
Let and F_2 be any two distinct members of . If either F^ or F_2 is A^ = $, then F^ and F_2 are automatically disjoint. If neither F^ nor F_2 is A^, then we can write
. „ . T n(l) . , „ T n(2). .
F 1 " X j(l) n A n(.l)+l. and F 2 = I j(2) n A n(2)+1'
If n(l) < n(2), then n(l) + 1 ^n[2),(so
p_2 = l"^2) n A n(2)+1 — A n(2) — A n(l)+1' and therefore Fj and F_2 are disjoint. A similar argument shows that if n(2) < n(l), then F^ and F_ are disjoint. Thus, if F_q arid F_9 are not disjoint, then n(l) = n(2) and we have
F 1 = I j(l) AA n+l 311(1 F 2 = ^(2)^^+!'
where n = n(l) = n(2). But then j (1) | j (2), so I^1?„. and I^3„,.. / J T J ' j (1)] (2)
split, and therefore F^ and F2 split. So we have shown that any two distinct members of either split or are disjoint.
If *p* has infinitely many distinct members, let E_q, E„, E„... be a counting of *p* . If has only finitely many distinct members, let E_n,..., E be the members of *'p* and let E, = d> for k > m. In either case, IE J is the desired sequence.Ⓡ
If F is a closed subset of the real line, then by a *complementary interval* of F we mean a component of F'. (If F = R, then $ is considered to be a complementary interval of F.)
Definition. By a *special family* we mean a family of subsets of R such that
(3) is nonempty
(4) each member of ^*p* is bounded and closed
(5) there exists a sequence' {F_nJ_n"j of members of *p* such that every member of *p* is equal to some F , and the following condition is satisfied.

(5a) If m > n, then either F_m is contained in one of the complementary intervals of F , or else F splits with F . n' m r n

Lemma 10. If E is an F^ set in R, then there exists a special family *jP* such that E = LJ .

• 00

Proof. By Lemma 9 we can choose a sequence $tE_n\}_{n=}i$ of bounded closed sets such that if n I m then E and E either split or are disjoint, 1 n m 47

00

and E = U E n=l

Let n. = 1 and let F. = E_n. Now suppose that n_q, n_9,..., n − t 11 1 s are chosen and F., F„,..., F are chosen so that 12 n s

(i) 1 = $n_T < n_2 < ... < n_s$

(ii) F. is closed and bounded (i = I,..., n) 1 o

(iii) if n_s >_ r > t >_ 1, then either F^ is contained in one of the complementary intervals of F , or else F^ splits with F^

(iv) if 1 £i £$^n{}_s$, then there exists j s} such

We construct F F as follows. Let be the family of

n_s+l ^{n}s+l

complementary intervals of the bounded closed set

We assert that E j meets at most finitely many members of xd^. If this **• co**

assertion is false, then there exists an infinite sequence $\{I_n)_{n\_}$^ of members of such that n m implies I_n<^ I = <|>, and there exists **' • '⊠ co**

(for each m) a point x € I HE {x} . is a bounded sequence, 47 m m s+1 m m=i

and n i m implies that x„| x . From this it follows that $\{x_m\}$ 1 r n 1 m m m=i

has either a strictly increasing or a strictly decreasing convergent

CO subsequence. We will assume that is a strictly increasing

convergent subsequence; the reasoning is similar

in the case of a

strictly decreasing convergent subsequence. Say 1*^ Ⓡ b^) .

^{X}m(k)^ we let

< X H X. m(k)

^{I}mCk+l)

< b, , so since x v < x , k' mfkj m(k+l)

^{b}k+l

we must have x „ < < a. . < x . m(k) — k+1 m(k+l)

and

Therefore, if

lim

k-*» ^{X}m(k)

then x

lim k-x»

Moreover, for k >_ *2*
finite real
number
so
that a. € \ J E•. Therefore there exists u €{1,..., s}
K £T1 1
such that a, € E for infinitely many values of k. Consequently KU.
x € E . But since x „ A £ E u m (.kJ
so that E and E
u
E and E .. u s+1 that are less
n must split s+1 r
Since infinitely
than x;
and E
, x € E . also. But then x €. E nE „ S + 1 U S + 1
and x must be a splitting point for
many
also
a^ lie in E^, E_u contains points contains points less than x;

therefore E and E .
cannot split
and we have a contradiction. This
u s+1
proves the assertion.
3 = {(J}u{InE . : 16^ and I n E . ± <>}. S+1 S+1
Let $n_{s+}i$ equal n_g plus the number of members of ,8. Let F_{n+}^,..., F be all the members Of We must show that conditions fi) ns+l

through (v) are still satisfied when s is replaced by s+1. Conditions (i)> $(ii)_f$ and (iv) are obvious. The verification of (iii) is divided into three parts. Suppose n_{g+}^ >. r > t >_ 1.

Case I. Assume that n$ > r > t > 1. In this case we already know that either F_r is contained in one of the complementary intervals of F_t or else F_r splits with F_t.

Case II.
Assume that n
. > r > n s+1 —

t > 1.

There exists v & {1
,s}

such.that F_t E_v» Either E_v and E_{g+}^ are disjoint or they split.

Case Ila. Assume E^. and E_{s+}j are disjoint. Either F*. = <j>(in which case F ' is certainly contained in a complementary interval of F) or else F 4 I and F = InE for some I€«j?. Let J be the smallest r 1 r s+1

— * i^S ⊠ '

closed interval containing F . Then J £ I and J £ I C (E.) , *so*

T • i 1

* 1=1

that J does not meet E . The endpoints of J lie in F <= E ., so v r r s+1'

neither endpoint of J lies in $E_{v>}$ So J does not meet E_v and therefore

J does not meet F ; from which it follows that F^ is contained in a complementary interval of F^.

Case lib. Assume that E and E . split. Since F. E and F £ E . v s+1 r t ~ v r s+1

it follows that F. and F split, t r

Case III. Assume that n , > r > t > n . If either F or F* is d>, s+1 — s r t T'

it is clear that F_r is contained in a complementary interval of F^.

Otherwise, there exist I,, I^ such that Ij A I2 = <l>and

F r = Z1 A E s+1 F t = *2nE s+r

Since 1^ and evidently split, F^ and F^ must split.

Thus condition (iii) is verified. /

As for (v), it is clear that , s V S+1|

E -U E.C2 F.CE q, s+1 1 i=n+l i s+1'

1=1 J s '

so that

Hence

Thus we

• ' co

have shown that we can construct sequences {n_.},._p

<v

co k=l

in such

a way that conditions (i) through (v) are satisfied

for every value of s. If we set^= {F. : k = 1, 2, ...}, it is easy

to verify that is a special

family and that E = ⊠

Definition.

If-J^ and are two families.of sets, let

^L a ^2 = {F i n

F 2 : F l^ ^1 F 2 ^2^'

Lemma 11. Ifandare two special families, then-J-A^is a special family.

Proof. Conditions (3) and (4) in the definition of a special family are clearly satisfied, so we just have to verify (5).

Arrange all pairs of positive integers in a sequence according to the scheme shown in Figure 1. Let (a(k), b(k)) be the kth term of the sequence''' (k = 1, 2, ...). Observe that k < *I* if and

^The reader may find it amusing'to derive the following formulas for (a(k), b(k)). For real t, let t denote the largest integer that is *strictly* less than t. Then

a(k) = $^{+}$ $_{+}$ ^{X}]]) - k + 1

= |(/8k71 + | - |(-1)^{C}(/8k+1 + | - |(-1) t['/8kU]]) _k

+1

Hctt/SkTTl] + 3) (MST + 1) - k + 1 if /81E+I is odd

I |('[/]8ic+T + 2)'[/]8lc+T - k + 1 if M+T. is'even, and

(1,2)
(1,3)
a $/☒ ☒
(2,1)
(2,2)
(2,31
(2,4) ☒ • •
Q,l)
(3,3)
(3,4) • • •
L4.1)
(+.2)
(4,3)
(4,4) • • •

Figure 1.

only if either a(k) + b(k) < a(£) + b(£) or else a(k) + b(k) = a(£) + b(£) and b(k) < b(£). Thus k < £ implies that either a(k) < a(£) or b(k) < b(£).

Let b$^{e\ a}$ sequence of elements of *f* such that every

member of^is equal to some F_n and such that condition (5a) in the

• "*☒ 00

definition of a special family is satisfied. Let $\{F_n)_n$_^ be a similar sequence for Set

^{F}k $^{=F}$a(k) $^{n\ F}$b(k) '

Then fF.K, is a sequence in such that every member of *J?* K K= JL

is equal to some F, . We must show that condition (5a) is satisfied.

Suppose that £ > k. Two cases occur.

Case I. a(k) < a(£).

Note that F^^ $^{F}{}_{a}$(k) $^{\text{and } F}$£ — ^{F}a(£) $^{\text{, Either } F}{}_{a}$(£) $^{\text{is contained in}}$ one of the complementary intervals of F . (in which case F. is contained in a complementary interval of F^), or else F^^ and $^{F}{}_{a}$(fc) split (in which case F. and F. split).

Case II. b(k) < b(£).

In this case a similar argument shows that either F_o is contained in

b(k) = |(2!®pJi _ /SF* *. -^{2}) $_{+}$ k

= 4(»<tF|] + I - i(-l) ^{[[,/}^^{1]}) ([[y^T - | - |(-1)U$^{\text{'/§E*T}}$^) +k

4(/8FT + l)(/8k?T - 1) + k

- 2) + k

if /8k+l is odd

if [[Vsk+l)] is even.

I

a complementary interval of F^ or F$ and F^ split. Thus condition (5a) is satisfied, and-J^A^is a special family. ☒

Lemma 12. Let E,, E_9 be two F , sets in R such that E. C E_9, and suppose that and are special families such that E^ = $^{\text{aIi}}$d ^2 = Then E^ = •

The proof is obvious.

Next we introduce some notation.

Let J be a nonempty interval on X with endpoints a, b (a £ b).

IT

By Trap (J, e, 0)(where 0 € (0, y) and e > 0) we mean the interior z

of the trapezoid shown in Figure 2. That is,

Trap(J, e, 0) = { (x, y> :0<y<e, a + y ctn 0 < x < b - y ctn 0}.

7T

For 0 g (0, ^-), let Tri (J, 0) be the closed triangular area shown in Figure 3. That is,

Tri (J, 0) = { <x, y>: y >_ 0 and a + y ctn 0 £ x £ b - y ctn 0}.

7T

If x € X, e > 0, and 0 € (0, y), let S(x , e, 0) denote the open 0 Z O

Stolz angle shown in Figure 4. That is,

S(x_o, e, 0) = { <x, y>: 0 < y < e, X_q + y ctn (it - 0) < x < x_q + y ctn 0).

If K is a closed set on a real line, let J(K) be the smallest closed interval containing K. If K is bounded, closed, and nonempty, e > 0, and 0 < B < a < p then we define

B(K, e, a, B) = Trap (J(K), e, a) - U Tri (I, B),

where denotes the set of complementary intervals of K.

x axis

Figure 2. — Trap(J,E ,0)

x axis

X_o
x axis

Figure 4.—S(x_0>£ ,9)

I > We state without proof the following readily verifiable facts.
(6) B(K, e, a, 3) is an open subset of H.
(7) S(e, e, 0) is an open subset of H.
(8) If Kjl and K_2 split, then for any e_2, *a,* 3, B(Kp Ej, a, 3) and B(K_2, e_2, a, 3) are disjoint. /
(9) Suppose that £ K, e > e^ > 0, and 0 < 3 < 3-^ < < a < y.
Then
B(Kp E_p (Xp 3p n H C b(K, e, a, 3) .
(10) Suppose Kj is contained in one of the complementary intervals of K, and suppose e, a, 3 are given. Then there exists 6 > 0 such that for every n £ 5,
B(K, e, a, 3) and B(Kp n, a, 8) are disjoint.
IT L *
(11) Suppose that a < 0 < and x_q J(K) . Then, for any e, Ep
B(K, e, a, 3) and S(x_q, E_p 0) are disjoint. ⊠ **7f**
(12, Suppose that x € K n J(K) and -3 < a < 0 < Let e be given.
Then there exists 6 > 0 such that for every h £ 6,
S(x_Q, n/ 6) *n* H S B(K, e, a, 3) .
(13) Suppose that e < e' and 0' < 0. Then
stx"TT^TAH £S(x_q, s', 0')-
(14) Suppose $x_q$$K and e, *a,* 3, 9 are given.. Then there exists 6 > 0 such that for. every h £ 6,
S(x_q, n, 0) and B(K, e, a, B) are disjoint.
(15) If x_q | x^ and e, 0 are given, then there exists 6 > 0 such that for every n £ 6,
S(x_q, e, 6) and S(Xp n, 0) are disjoint.
(16) B(K, e, a, B) n X C K.
(17) S(x_o, e, 6) nx = {x_o}.
2
Definition. If^ris a special family, let T be the set of all members of *7* that have two or more points.

Definition. LetJ^be a special family, let E be the set of all endpoints of intervals J(F) where F € F , and suppose that O<B<a<0<H-. By a *pair of special a,* 3, 0 *functions* for we mean a pair (e, 6), where e and 6 are positive.real-valued functions, 2 the domain of e is E, the domain of 6 is , and
(18) for each n > 0, there exist at most finitely many F € such that 6(F) >_ p;
(19) for each n > 0, there exist at most finitely many e e E such that e(e) £ n;
(20) if e, e* € E and e e', then
S(e, c(e), 0) and S(e', E(e'), 0)

are disjoint;

(21) if F, K ef^2 and F 4 K, then

B(F, 6(F), a, B) and B(K, 6(K), a, B) are disjoint;

(22) if e € E and F e^^{2}, then

S(e, e(e), 0) and B(F, 6(F), a, B) are disjoint.

Lemma 13. Letbe a special family and suppose that Oc'B<a<0<y. Then there exists a pair of special a, B, 6 functions for

Proof. Let {F_n}$_T$7j be a sequence of members of 5^{s}"of the type referred to in condition (5) in the definition of a special family. Let (J^(n) = {F € J®^{2} : F = F^ for some k <_ n}

E = set of all endpoints of intervals J(F) for FGf, F 4 I

E(n) = {e € E : e is an endpoint of J(F^) for some k £n for which F^ 4

If J(Fj) has one endpoint e, set e(e) = 1. If J(F^) has two endpoints e^, 62, then by (15) we can choose e(e^) £ 1 andc(e_2) £ 1 so that S($e_{1\}}$ e^), 0) and S(e_2, ^{e}C^{e}p> ^{9}) are disjoint. If $F_1$6'f^2, set 6(F_1) = 1. In this case, J(F_n) hats two endpoints e. and e_9 and (by (11)) B(F_1, 6(F_1), a, B), S(e_p e^), 0). and S(e_2, cfep, 0) are *1* all disjoint.

Now suppose that e(e) and 6(F) have been defined for all

2

e € E(n) and all F Gy (h) in . such a way that

(i) if e, e' € E(n) and e | e^1, then S(e, e(e), 6) and /

S(e',e(e*), 6) are disjoint;

(ii) if F, K€dP^2(n) and F | *K,* then B(F, 6(F), a, B) and B(K, 6(K), a, B) are disjoint;

2

(iii) if e €E(n) and F€(p (^{n}) > then S(e, e(e), 0) and

B(F, 6(F), a, B) are disjoint;

(iv) if e € E(n) and k <nis the least integer for which

e *G* E(k), then e(e) £

2

(v) if F £ jF (n) and k < n is the least integer for which F€f*(k), then 6(F) <p

We must extend the definitions of e and 6 to E(n+1) and

2

tF (n+l) in such a way that conditions (i) through (v) are still satisfied when n is replaced by n+l.

2

Case I. If F . = * or if F . = F. for some k < n, then J^1 (n+l) = n+l n+l k — ^{v}

^(n) and E(n+1) = E(n), so that nothing is required to be done.

Case II. If F , consists of a single point e and if e 6 F, for some n+l k

k < n, then (since F . and F. must split in this case) e is an n+i k

endpoint of J(F.), so that again 7^{s2} (n+l) = ^p^2(n) and E(n+1) = E(n), K

and nothing is required to be done.

Case III. Suppose that-F j consists of a single point e_Q and that for each k < n, e F-. . By (14), (15), and the fact that E(n) and

(n) are finite, we can choose $e(e_o)$ €£ (0, so that $S(e_Q, e(e_Q), 0)$ is disjoint from S(e, e(e), e) and from B(F, 6(F), a, B) for each e €E(n) and each F^^pfn). The construction is then finished for

E(n+1) and *F*(n+1).

I

Case IV. Suppose that $F_{n+}j$ contains at least two points and that, for each k < n. F, i F ,. For each k < n, either F . splits with F. , or else F_{n+1} is contained in a complementary interval of F^. Since ^(n) is finite, (8) and (10) show that we can choose $^5(F_{n+1})$ £(0, ^y) so that $B(^F{}_{n+}i, {}^5({}^F{}_{n+}\hat{}), a, 6)$ is disjoint from B(F, 6(F), a, B) for each F€j*(n).

Say e € E(n). Then e is an endpoint of $J(F_k)$ for some k <_ n, so (since F_{n+}^ either splits with F^ or is contained in a complementary interval of F_R) e $\hat{}J(F_{n+1})^*$. By (11), $B(F_{n+1}, 5(F_{n+1}), a, 8)$ and S(e, e(e), 0) are disjoint.

' Let e , e [1] be the endpoints of J(F $_n$).

o o ^{r} n+1

Case IVa. e_Q, e • £E(n). ,

In this case the construction is already finished.

Case IVb. e_Q€E(n) and e_Q* $E(n).

If e ' £ F. for some k < n, then F , splits with F. , so that o k — ' n+1 ^{r} k'

e_Q' must be and endpoint of J(F^) –which contradicts the assumption that e ' 4 E(n). Hence, for each k < n, e ' £ F. . By (14), (15), and the fact that E(n) and J^(n) are finite, we can choose

1 — -

e(©$_0$') C (0, ^-y) so that $S(e_o', e(e_o')$» 0) is disjoint from

S(e, e(e), 0) and from B(F, 6(F), a, B) for each e6 E(n) and each F€ $f^2(n)$. By (11), $S(e_Q', e(e_o')$» 0) and $B(F_{n+1}, 6(F_{ft+1}), a, B)$ are disjoint. Thus the construction is finished for E(n+1) and(n+1).

/

Case IVc. e_Q E(n) and e * € E(n).

This case is essentially the same as Case IVb.

Case IVd. e_Q E(n) and e_Q'

<E(n).

k < n. then F y-snlits with F, . so

If e 6F, for some o K

e_Q is an endpoint of $J(F_k)$; a contradiction. Thus e^ F_k for k £ n, and similarly e ' F. for k <_ n. Therefore, by (14) and (15), we can choose e(e) and e(e ') E (0, —rr) so that S(e , e(e), 9) and o o n+1 o o

$S(e_o', E(e_o'), 6)$ are disjoint and each of $S(e_Q, e(e_Q), 9)$ and

$S(e_o'j\ ^e(e_o')>\ ^0)$ ^{is} disjoint from every S(e, e(e), 9) (e €E(n)) and ~ from every B(F, 6(F), a, 8) (FG'J^n)). By (11), $S(e_Q, e(e_Q), 9)$ and

$S(e_o', e(e_o')>\ ^0)$ ^{are} each disjoint from $B(F_{n+1}, {}^a$» > $^{so\ t}$^e construction is finished for E(n+1) and(r (n+1).

I

We have shown that we can inductively define e(e) for every e € E and 6 (F) for every F € f^2 in such a way that (i) through (v) are satisfied for every value of n. Conditions (20), (21) and (22) in the definition of a pair of a, B, 9 special functions are thus automatically satisfied by (e, 6). We must verify that (18) and (19) are also satisfied.

Suppose (19) is false. Then there exists n > 0 and there

00 exists an infinite sequence distinct members of E such that

E(^{e}k) j£ n for every k. Let m(k) be the least integer for which e^ is an endpoint of J(F ,...). Each J(F) has at most two endpoints, so, m j in [1]

since the e^ are all distinct, there exists (for given m) at most two values of k for which m(k) = m. Therefore there exist infinitely

many distinct integers among m(l), m(2), m(3), ... Consequently

there exists j with [J] m(j)

< n. But, by (iv), e(e..) £ ^

contradiction. So (19) must be true. A similar argument shows that

(18) is true. ⊠

_IT

Lemma 14. Let *T* be a special family, 0 < g <. a < 0 < y, and. let E be the set of all endpoints of intervals J(F) for F €-J*. Suppose (e, 6) is a pair of special a, g, 0 functions for If e^, are two real- 2

valued functions having domains E and^f respectively, and.if

0 < E|(e) £ e(e) for all e £E, and

0 < £ 6(F) for all F^J^2,

then (Sp 6^) is a pair of special a, 0, 0 functions for^.

Proof. This follows from the fact that

$S(X_o$, E', 0) <^S(X_o, E", 0) I

and B(K, e', a, g) OB(K, e", a, g)

[1] whenever e' £ e". 0|

Theorem 4. Let A be any set of type F_{ag} in X. Then there exists a . bounded continuous complex-valued function f defined in H such that A

is the set of curvilinear convergence of f.

co

Proof. We can write A = f | A , where each A is of type F and ' J n' n *o*

n=l

A_{p+1}^A_n for every n. For each n, let be a special family with CX-v [ut]

-31

" ^n[A] ^n+l [for n] '

By Lemmas 11 and 12, together with mathematical induction, *K* is a special family and [= A]n- Moreover, every member of^$_+$^ is a subset of some member of^.

00

Let be a strictly ascending sequence in (0, $-)
. IT
converging to g-.
Let {<x be a strictly descending sequence in ($-, ⊠$) • 11 converging. to ©o 7T ' '
31T
Let be a strictly ascending sequence in Q, •—)
converging to g—.
Let E be the set of all endpoints of intervals J(F) for
Let (e(l,0, 5(1,-)) be any pair of special o^, 8^, 6_1 functions for .
Nov; suppose that for each k<nwe have chosen a pair of
special a, , 3„ 6, functions (e(k,-), 6(k,-)) for *ft.* in such a way K K K i*
that
' (i) whenever 1 £ k £ n - 1, eg $^{E}k_{+}$p F g
e € F n J(F) , then
S(e, e(k+l, e), 6_{k+1}) AHClW 6(k, F), <x_k, 3_k);
(ii) whenever 1 £ k £ n * 1, eg
., and e g E. , then k+1* k'
SCe, e(k+l, e), 6_{k+1}) nHCs(e, e(k, e), 6_k);
(iii) whenever 1 £ k £n - 1, Kg-$J^{\wedge}{}_{+1}$, Fe$^{\wedge 2}$, ^{K} — ^{F}» then
B(K, 6(k+l, K), a_{k+1}, B_{k+1}) AHCB(F, 6(k, F), a_R, 3_k).
Then we construct (e(n+l,-), 6(n+l,')) as follows. Let
(e, 6) be any pair of special a_{n+}p $^{\wedge}{}_{n}$+l' ^{9}n+l $^{:Funct}$l°ns ^°^^n+i' ^-2 *
e g E_{n+}i - E_n, then for some unique F€ , e £ F A J(F} , so by (12) we can choose 5(e) >0 such that n £ 5(e) implies
S(e, n, 0_{n+1}) nHGB(F, 6(n, F), a_n, $\&_n$).
We set e(n+l, e) = min {c(e), 5(e)). On the other hand, if e g E_{n+}^ *r*| E^, then we set e(n+l, e) = min {e(e), e(n, e)}.
If then there exists a unique K with F £ K.
Set
6(n+l, F) = min {6(F) , ~ 6(n, -K)}.
By Lemma 14, (e(n+l,*)> <5(n+l,')) is a pair of special a p
B ., 6 , functions for^.„ and by (13) and (9), conditions (i), /
n+1 n+i n+i (ii), and (iii) are still satisfied when n is replaced by n+1. Thus we can inductively construct a pair (e(n,*)> 6(n,*)) of special a_n, $\$_n$, 0_n functions for in such a way that conditions (i), (ii) and
(111) are satisfied for every n.
Let
V_J ^{S}O> ^{e}(^{n}» ^{e}) > ^{0}J | V e € E„ J
VL ^{B}C^{F}> ^{6}<^{n}> ^{F})"' € t n n
Then U_n is open. For fixed n, all the various sets S(e, e(n, e), 0^) (e € E) and B(F, 6(n, F), a_n, $\&_n$) (F € 5^) are open and pairwise disjoint, so that every component of U_n is contained in one of the sets ^,2

S(e, e(n, e), 0_n) (e € E_n) or B(F, 6(n, F), «$_n$, &$_n$) (Ffify. It therefore follows from (16) and (17) that if Q is any component of U , then

(23) « AX A_n.

From the fact that (e(n,'), 6(n,*)) is a pair of special a_n, B_n, ©$_n$ functions for 5^ together with conditions (18) and (19), it follows that

u_n^{nH} = tU S(e,.e(n, e) , 6) HH] u e 6 E n ....

[U_2 B(F, 6(n, F), a_n, y n H].

Consequently, conditions Ci)> (ii), (iii), together with the fact that e € E , - E e 6 F n *J(F) for some* [1]

n+1 n [1 J] n'

show that U , H U for every n. n+1 n

By Urysohn's Lemma, there exists a continuous function

g_n : H ⊠+ [0, 1] such that

$g_n(z)$ = 1 for z € H - U_n and g (z) = 0 for z €. TT' _ n H.

⊠ n n+1

co

Let g(z) = S — s (z). Then 0 < g(z) <_ 1, and the series converges n=l 2^n

uniformly, so g is continuous on H.

If

z € H - U_n, then z £ H - for every m >_ n

so

that

and hence

(24)

CO

. g(z) 1 E m=n

1 = 1

2^m 2^{n-1}

(z € H - U_n) .

Also

if z € U „ n+1'

then z € U„ U», ,.., U „ so that 12 n+1

= $g_n(z)$, and

(25)

. i z ns m=n+l 2,u

1

2^n

€ U n+1

We assert that

3ir

(26) for each x_q € A, g(z) + 0 as z + x_q with z € S(x_q, li- g—) •

Take any natural number n. Since x C A , = i> either 7 o n+1 v[†]z*'n+l' case, set n = e(n+l, x_q). In the.second case, (12) shows that we can choose n > 0 so that

S(x_q, ni J^1-) (F, 6(n+l, F), a_{n+1}, .

Suppose <x, S(x_q, 1, -*g*-) and y < n. Then, in the first case, <X, y>€ S(x_o, n, ⊠§-) £S(x_q, e(n+l, x_q) , $6_{n+]}$) ^U_{n+1}, and in the second case,

<x, y>£ S(x_o, n, £b(F, 6(n+l, F), B_{nU}) £ U^. So, by (25), (<x, y>6 S(x_o, 1, jl) and y < n) =><x, y>£ U_{n+1}

0 5 g(x, y) 2^{11}

This proves (26). [1]

Let x_q be a point in X and y any arc at x_q. Suppose g(z) ⊠> 0

as z ⊠* x^ along y. Then y has a subarc y' with one endpoint at x_q

such that y* - {x_q} Cg"'l((- Therefore, by (23), x_q € A_{fl}.

1_ £.

2^{n}' 2^n Since

0). By (24), y' - {x^.

n was arbitrary, x_q

[1] i n n=l

= A.

Thus,

(27) if there exists an arc y at x_q such that g(z) •* 0 as z approaches x_q along y, then x$ € A.

Now define

f(x, y) = . gCx, y) sin + i g(x, y) (<x, y>€ H).

3tt

If x_q e A, then, by (26), f(z) •* 0 as z -> x_q with z GS(x_q, 1, •$—).

Thus every point of A is in the set of curvilinear convergence of f.

Conversely, suppose x_q is any point of the set of curvilinear

convergence of f. Let y be an arc at x_q such that f approaches the limit c + di along y. Then g approaches the limit d along y. If d is different from zero, then g(x, y) sin — (the real part of f) cannot approach any limit along y — a contradiction. Therefore g approaches the limit 0 along y, and, by (27), x_q € A. Therefore A is the set of curvilinear convergence of f.M

5. Boundary Functions for Continuous Functions

Lemma 15. Let E be a metric space, Y a separable metric space. *i*

Suppose that : E -> Y is a function having the following property. For *every* open set U $=Y there exists an F set L?E and a countable set NO such that

9^{-1}(U) CL C <p^{-1}(U) u N. ~"

Then there exists a countable set M ^E such that cpL is of class e-M

CF_a (E - M)).

Proof. Let B be a countable base for Y. For each B€&, let L(B)
be an F^ set and let N(B) *Q* E be a countable set such that
$<^{1}$(B) *Q* L(B) C $<p^{-1}$(B) u N(B)
Let M = VJ N(B). Then M is countable. Let E = E - M and let B€fi °
= <p|g . We show that $<p_o$ is of class (F (E_q)) . o .
Let W be any open subset of Y. If p € W, there exists r > 0 such that S(r, p) *Q* W. Choose B€ Ⓡ so that p €BSs(| r, p) . Then JBSS(r, p) Cl w. It follows that
N = U B = U
B€dt(W) B € d(W)
where (J.(W) = £B€Ⓡ : B" ^W}. Therefore
V_b^{-1}(W) = E A 'p'M = E A VJ • $<p^{-1}$(B)
0 0 0 B€ffl(W)
- E A U L(B) ^{0} Bcacw)
— E U [$<p^{-1}$CB) u N(BJ]
B£ (B(W)
CEO O [CP[‡]Ⓡ U M]
B€Q(W)
= e n U $<p^{-1}$CB) ° b *e* <j(w) -i -i
= E_on$<p^1$CW) = *q* ^{1} (W). o
Consequently cp^CW) = E_q H $_b$ -L(B), so ^"d(W) is of class W'^{1}
Theorem 5. Let Y be a separable metric space and let f : H •> Y be a continuous function. Suppose that E C X and that cp : E -> Y is a boundary function for f. Then there exists a countable set M *Q* E such thatcpL .. is of class (F (E - M)).
Proof. Let U be any open subset of Y, and let W = (U)'. Set
E_n = {x € X : there exists an arc *y* at x, having one endpoint on X_n, such that *y* - {x} -f'\u)}
K = {x € X : there exists an arc *y* at x such that
Y - {x} C f"^{1}(w)}. '
Ob serve that
^(U) cQ E n=l ^{n} and <jp–^{1}(W) *Q* K.
For the time being, let n be a fixed natural number. For each x € K we can choose an arc y_x at x such that
Y_x - {x} £ H_n n f^{-1}(W) .
Since an arc.at x is by definition a simple arc, y_x - {x} is a connected set and hence must be contained within one nonempty component of H n f"\w). Let U denote this component (for each x € K) . n * x
Let T be the set of all points of K that are two-sided limit points of $E"_n$. We claim that if x, y € T, then x | y implies
U_x n U = 4>. If U_x n Uy. <j>, then (since U_x and U^. are two components
of the same set) U and U are equal. Let p be the endpoint of y x y x
lying in U and X

let
q be the endpoint of y lying in U^. = $U_{x<}$
We can

join p and q by an arc y lying in U . X
Putting y , y and y together, x y
we obtain an arc a with one endpoint at x and the other at y, such that a - {x, y}gU'. According to [12] we can choose a simple arc
a' £ a having one endpoint at x and the other at y. Of course, a' - {x, y) $^{u}{}_{x}$^ A f^{-1}(W). Let I be the open interval in X with endpoints at x and y, and let J = X - I. Let B be the bounded component of H - a' and let A be the other component. Since X_n is unbounded and does not meet *a*', X £ A. n
Because x is a two-sided limit point of E_n, we can choose a point w €. I H E_n« Let g be an arc at w, having one endpoint on X_n, such that g - {w} C f'^CU). Then g does not meet a' (because a' - {x, y] ^.f \w) and f \w) f \u) = <f>), and therefore (since g. - iw} contains a point of X C A) g - {w} £. A. It follows that
w € A. This, however, is a contradiction
because the frontier of A
(relative to the finite plane) is a'u J.
We conclude that, for x,y € T, countably many components,
x y implies U*. *C*\ U = <j>.
An open set in the plane has only
so it follows that T must be countable. Let S be the set of all
points of E_n that are.not two-sided limit.points of $E_{n>}$
.We know that
S is.countable, so
K A E_n = [K n (E - S)] U [K n S]
= T u [K aS]
is countable. 00 ⊠" ⊠
Let N = K *n* \ ^*J* E = (K HE). Then N is countable, and, . n=l n=l
since <p (W) K, . -
00 00
<p^{-1}(U) *Q* Ea|J E C E
n=l n=l
00 CO
= (E A K E) U ((E - K) n E) n=l n=l
(E n N) U (E - K) C (E nN) U (E - (p"1 CW))
= (E A N) U <p"1(U) .
1. 00
Thus <p"l(U) *Q* E A \ ^_ *J* E C(EnN) u <p""'(U), and the desired result n=l
follows from Lemma 15. ⊠

Corollary. Let Y be either the Riemann sphere, the real line, or a finite-dimensional Euclidean space. If f : H + Y is a continuous function, if E *Q* X, and if q>: E + Yis a boundary function for f, then *cp* is of honorary Baire class 2 (E, Y).

Next we show that the foregoing corollary is as strong as possible in the sense that if E is any subset of X and (p is a function of honorary Baire class 2 mapping E into a suitable space, then there exists a continuous function in H having f as a boundary function. A proof of this result– at least for real- or vector-valued functions was outlined by Bagemihl and Piranian [2, Theorem 8], in the case

where E = X. Although the construction given here is carried out much *l*

more explicitly than the construction given by Bagemihl and Piranian, my treatment differs from theirs in only two aspects that are of any significance. First of all, the proof of the theorem for arbitrary subsets E of X depends on Lemma 6 of the Introduction. Secondly, Bagemihl and Piranian say in the last line of their proof that there is "no difficulty now in extending f continuously to the whole of D in such a manner that <j>is a boundary function for f." While this appears to be all right for real- or vector-valued functions, it is not clear why the extension should be so easy for functions taking values on the Riemann sphere. Theorem 7 of the present paper shows, however, that the result can be obtained for functions taking values on the sphere once it is known for vector-valued functions.

The following miniature closed graph theorem will be a convenience.

Lemma 16. Suppose that M is a metric space and that u : M -> R is a function having the following properties:

(i) if $\{p_n J$ is a convergent sequence of points of M, then $\{u(p_n)\}$ converges neither to + » nor to -

(ii) if {p } Cm, p € M, and y € R, and if p -⊠ p and n n $_n$

u(p_) ⊠* then u(p) = y.

^{n} n Then u is continuous.

Proof. Suppose that {p^} is a sequence of points in M converging to a point p € M. Using (i) it is easy to show that $\{u(p_n)J$ is a bounded sequence. Suppose that $\{u(p_n)\}$ does not converge to u(p). Then there exists a subsequence {u(pthat converges to a real number y u(p). This, however, contradicts (ii). We conclude that

u(p_) ^{u}Cp) • ⊠ •

n

Lemma 17. Let h : R + R be a strictly increasing function such that h(R) is neither bounded above nor bounded below. Then there exists a *1c* ±

continuous weakly increasing function h ; R ⊠* R such that h (h(x)) = x for every x € R.

Proof. Let Z = h(R). Observe that h'^{4} : Z ⊠> R is strictly increasing. For any x € R, the set (-^{00}, x] *r*| Z is nonempty. Also, h$^-$\(-°°, x] n Z) is bounded above, because if we choose y G Z with x ^y, then

-1 -1 h ((-"> x] Z) is bounded above by h (y) .

We claim that for every x *E* R

-1 -1

(27) sup h ((-«>, x] n Z) = sup h ((-<», x) n Z).

If x £ Z, the equation is trivial. Suppose x G Z. Then y < h"\x) (h(y) < x and h(y) € Z), .

so that h((-«>, h"^(x))) C (-oo_J x) n Z. Hence

(-«, h^4(x))Ch^4((-», x) n Z),

so that sup h^{-1}((-<», x) n Z) >_h^{-1}(x) = sup h^{-1}((-«>, x] n Z). The opposite inequality is trivial, so (27) is established.

We also claim that

(28) inf h~'l((x, +°°) rv Z) = sup h^{-1}((-°°, x] n Z).

-1 -1

Obviously, inf h ((x, +")nZ) >_ sup h ((- <», x] n Z). Take any y >sup h \(-<», x] n Z). If h(y) x, then h(y) £ (-~, x] n Z, and

so ygh ((-«>, x] n Z)– a contradiction. Thus h(y) > x and h(y) € (x, +") n Z. Therefore y^h"^((x, +~) n Z), and so inf 'h 1((x, +°°) *n* Z) _< y. In view of the choice of y, this implies

that

inf h^{-1}((x, +") n Z). £ sup h"1 ((-00, x] n Z), and (28) is established.

Define

* -1 h (x) = sup h ((-<», x] n Z).

* [§]

It is clear that h is weakly increasing and that h (h(x)) = x for ' * *I* every real x.. The continuity of h can easily be deduced from the equations

"k ft

sup h ((-», x)) = h (x) inf h*((x, +«)) = h*(x),

which are established as follows:

sup h*((-~,.x)) = sup sup h"1(C-°°, y] *r*\ Z)

= sup h~1((-«% x) n Z)

= sup h"l((-", x] *rx* Z)

, * ⊠

= h (x)

inf h ((x, +«)) = . sup h ((-<», y] n Z)

= juf ^f h–1((y, +») *rx* Z)

= inf h~\(x, +«) *rx* Z)

= sup h"\(-°o, x] n Z) ,

* *

= h (x). ⊠

Theorem 6. Let E be any subset of X and let <p: E ->be any function of honorary Baire class 2(E, R^). Then there exists a ontinuous function f : H ->R^ such that cp is a boundary function for f.

Proof. Let ip : E ⊠> be a function of Baire class 1(E, R^) and N a countable subset of E such that <^(x) = ip(x) for every x € E - N. Let {s } . (with n + m implying s

4 s) be a countable dense subset of X that includes every integer and every point of N. Let

t =1 if s is an integer

t = — if s is not an integer,

n 2ⁿ ' n 6

Define

h(x) = E t if x > 0

$0<s_n<x$ n

i. h(x) = .-E t if x £ 0.

x<s <0 — n—

Then h is a strictly increasing function from R into R, and h(R) is

* bounded neither above nor below. Let h be the function described in Lemma 17.

Suppose that 0 < y <1. Then (for fixed x)

*, x - (l-y)u , _ „

u - h (———————— -

is a strictly increasing continuous function of u that approaches +°° as u -> +~ and -» as u -> -<». Consequently there exists precisely one number u(x, y) that satisfies the equation

(29) u(x, y) - h*(y)) = 0.

I claim that u(x, y) is a continuous function on

= { <x, y>: x, y € R and Q < y < 1}. Suppose' { $<x_{n>}$ y^} C H_1

and $y_n>$ * <*, y>€ H_r If u(x_n, y_n) + then - *xn - >'n)* _ .

y_n n '

and hence

* x - (1-y)u(x , y) u(x , y) - h*(n n >

n' 7n' k y which.contradicts (29) , Thus u($x_{n?}$ y) cannot approach +»., A similar argument shows that u(x_n, y) cannot approach -w.. Now assume that

u(x n

Then, by (29)

lim r *r* -s k* *r*

n-x» tu(xn' C

x: -r. (1-y)u(x , y)

* x - (l-y)u

- h (—————————- —— -

so u o

= u(x

y) •

By Lemma 16, u is continuous.

From

Lemma 6, there exists a sequence °f continuous
functions mapping X into such that g^(x) •* ip(x) for each x € E. n
For n > 2
define
$f_o(x, y) =$
$(yn(n+l) - n)g_n(u(x, y)) + ((n+1) - yn(n+]))g_{n+]L}(u(x, y))$
☒ 1 1 when —=- < y < —. n+1 — 7 — n
Then f is o
continuous
on
we can assume that
is
$^r n$
o
inf x>s
n
2 " "•
defined
h(x)
By the Tietze extension theorem
and continuous on
all
of H. Let
$^v n$
sup x<s n
h(x)
$V(^s{}_n) - Ks_n)$
if s € N n
$^v n$
if s £ N. n T
If x
and y are real numbers, define
x V y = max{x
y>-
set
$A_n(x, y) =$
[(1 - ny) V 0] [(1 - ——^{1} . X A/
n n
I r + A - 2s
^{1} n n n
. .s -x
2 -2— y

Then

A_n

is continuous in H. Observe that $A_n(x, y) = 0$ when y >_

Using this fact, it is:easy to show that, if we set

f = f + E A , 0 n=l n

then f is defined and continuous in H. We now show that (pis a boundary function for f.

Let p be any point of 'E. The line

(30) x = (h(p) - p) y + p

passes through (p, 0), and the part of it that lies in is an arc at p. We will show that f approaches ip(p) along this line. If we substitute (h(p) - p)y + p for x in the expression for A (x, y), we i ⊠ n

obtain

(31) $A_n(x, y) =$

[(1 - ny) V 0] [(l- —i-j- |r_ <• 1 4 2(1 - 1) (s - p) n n 3

- 2h(p) |) V O] $v_{n<}$

If p £ s , then h(p) £ $\&_n$, and one can verify directly that (31) vanishes. If p > s_n, then h(p) £ r , and again one can verify directly that (31) vanishes. Thus $A_n(x, y)$ vanishes along that part of the line (30) lying in H.

Solving (30) for h(p), we find that, along the given line, h(p) = ,

and hence p = h (h(p)) = h (-—-Xl-XlE) . Therefore, if 0 < y < 1, p = u(x, y) . So, if ^x, y^ satisfies (30) , n £ 2, and £ y £ ~, then

$f_0(x, y) = (yn(n+l) - n)g_n(p) + ((n+1) - yn(n+l))g_{n+1}(p)$.

Since the coefficients of $g_n(p)$ and $g_{n+}j(p)$ in the above expression add up to 1 and since both coefficients lie in [0, 1], f (x, y) lies on the line segment joining g_nCp) to g_{n+}^Cp)» an^ it follows that $f_Q(x, y)$ approaches ip(p) as y^ approaches p along the line (30). this line lying in H, f(x, y) show that f approaches (s^) that lies in H. Again, we first consider the value of along the

Since each A_n vanishes on the part of also approaches ip(p) along the line.

Let s^ be any point of N. We along the part of the line r. A

(32) x = (——— x—- - sjy +

given line. Substituting the value of x given by (32) into the expres

sion for A , we obtain n'

(33) $A_n(x, y) =$

[Cl-ny) V 0j[(l - -1- $|r_n - r_{m++}$ 2(1 - 1) (s_n- |) V 0]v_n.

n n

If s < s , then £ < r < £ < r , and one can verify directly that m n m m r-. n n'

(33) vanishes. If s < s , then £ < r < £ < r , and again one can n m n n — mm

verify that (33) vanishes. Thus, for n | m, ^(x, y) = 0 when <x, y^ lies on the line (32) and in H.

If we take n = m in (33), we obtain

$A_m(x, y) = [(1 - my) V 0]v_m$.

Therefore $A_m(x, y)$ approaches $v_m = 9(^s{}_m)$ - along the given line. Take any <x, y>€ satisfying (32), and take any a and b satisfying

(34) a < s < b.

m

⊠^ < r^ £ h(b), so that

Then.h(a). < ft . < - — m

(h(a) - s)y + s m ' m

x < (h(b) - s)y + s ; from which we deduce that m m

h(a).

x - (l-y)s

*

Since h

is weakly increasing, * * x - (lry)s

: h (h(a)) <h (————- ——

m *

—). <h (h(b)) = ,b.

Since a

and b were

taken to

be any two numbers satisfying (34), we

conclude that

whence it follows that u(x, y) =

s . Thus m

$f_O(x$>

y) = (yn(n+l) - lOg^s^) +

when

$f_O lx$

1 1

(x, y> lies on the given line and £ y £ y- Consequently y) approaches $ip(s_m)$ along the line (32). So f approaches $<p(s_m)$ +

<p(s) ' $^(^s{}_m)$ = $<p(^s{}_m)$ > and the theorem is proved. ⊠

2

Theorem 7. Let E be any subset of X and let cp: E -> S be any

2

function of honorary Baire class 2(E, S). Then there exists a

2

continuous function f : H -> S such ithat <p is a boundary function for

f.

Proof. The proof of this theorem is very similar to that of Theorem 1.

2 3
Since S CR , there exists, by Theorem 6, a continuous function
3
g : H -> R having ipas a boundary function. Let
K = g^{-1}({v € R^3 : |v| = | J)
L = g^{-1}({v € R^3 : |v| > | })
F = g^{-1}({vGR^3 : |v| £y }).

Let g_0 = gl^. H is homeomorphic to R , so by [5,.Lemma 2.9,p. 299],- g can be extended to.a continuous function ^{s}o
g_1 : H -> {v G. R^3 : | v | = . •
3
Define f^ : H -> R - {0} by setting <
fjCz) = g(z) if z € L
f_x(z) = . g_x(z) if z € F.
Then, since F and L are closed, f^ is continuous on H. It is easy to 3 2
verify that <p is a boundary function for f^. Let P_q : R - {0} ⊠* S
2
be the 0-projection onto S (see page 11), and let f be the composite
2
! function P_Q « f^. [1] Then f maps H continuously into S , and P_o« <p =
is a boundary function for f. ⊠

CHAPTER II. BOUNDARY FUNCTIONS FOR DISCONTINUOUS FUNCTIONS

6. Boundary Functions for Baire Functions

It is not known whether the set of curvilinear convergence of a Borel-measurable function defined in H is necessarily a Borel set. The answer is not known even for functions of Baire class 1. However, a theorem on boundary functions that is similar to the corresponding result for continuous functions in H can be proved for functions of Baire class 5 in H.

Definition. If A and B are two sets, we will call A and B *equivalent* and write A - B if and only if A - B, and B - A are both countable. It is easy to check that - is an equivalence relation.

Lemma 18. If A s E, then S - A - S - E for any set S. If A - E ' ' n n
for all n in some countable set N, then
LJ A ^{s} Ij E and p| A " P| E . n 6N n £ N n € N n e N
The proof of this lemma is routine.

Definition. An interval of real numbers will be called *nondegenerate* if it contains more than one point.

Lemma 19. Any union of nondegenerate intervals is equivalent to an open set.

Proof. Let be any family of nondegenerate intervals. .It will
suffice to prove that Ui-UI is countable. We can is4J- ie<5-
write

I€&
-*J* n * n
where {J } n
is a
countable family of disjoint open intervals.
If
then x o
so that
is
an endpoint of I
for some I € For some n, I o
n*
* I
is an endpoint of J^.
*—
G J o n
Thus
is
contained in the
set of all endpoints of
the
various J . and the n'
lemma is
proved.

Lemma 20. Let h be a weakly increasing real-valued function on a nonempty set E £r. Suppose that |x - h(x) |. £ 1 for every x € E. Then h can be extended to a weakly increasing real-valued function h^ on R.

Proof. Let e = inf E (e may be -»). For each x € (e, +“), set

h^x) = sup h ((-<», x] *r*\ E).

Since |t - h(t) | 1 for each t € E,

t G (-°°,; x] n E r^>h(t) — *x* + 1,

so h^ is finite-valued. If e = - <» we are done. If e >then x e E implies h(x) >x - 1 >e - 1, so h is bounded below. For x C C-°°, e] set

h^(x) = inf h(E).

It is easy to verify that h.^ has the required properties. ☒

Lemma 21. Let Y be a metric space, f : R •* Y a function of Baire class 5(R, Y), and suppose that h : R -> R is weakly increasing. Then there exists a countable set N Cr such that the composite function f o $h|_D$ .. is of Baire class 5(R - N, Y). K-In

Proof. Let N be the set of discontinuities of h. By a well-known theorem, N must be countable. But then $h|_R$ is continuous, so that f ® ($h|_{R\ N}$) = (f <» h) $|_{R\ N}$ is of Baire class ?(R - N, Y). 8

Lemma 22. Let Y be a separable arcwise connected metric space, E any metric space, and let <p: E ->Y be a function having the following property. For every open set U *= Y there exists a set T € P^+\e) such that <p^{-i}(U) Ct Then, if £ >_2, <p is of Baire class 5(E, Y) .

Proof. The proof is similar to that of Lemma 15. Let (B be a countable base for Y, and suppose that W is any open subset of Y. Let

d(W) = {US® : U C $_W$}.

The argument in the proof of Lemma 15 shows that w = u = L7 u. U€(J(W) U€O(W)

For each U€(fi, let T(U) € P^+\e) be chosen so that «p'^{1}(U)^T(U)G<p'^{1}(ii). Then

<p^{-1}(W) = T(U)

ued(w) ^T uedcw)

U€d(W)

Thus <p \w) = T(U) , and since P^+\e) is closed under countable

uedOD

-1 E+1

unions, <p (W) € $P^{5,}$ (E) . Therefore is of Baire class £(E, Y) . ☒

Theorem 8. Let Y be a separable arcwise

f : H + Ya function of Baire class g(H

X, and <p: E ->Y a boundary function for

C + 1(E, Y).

Proof.

Observe

that

connected metric space,

Y) where

f. Then

Let U be any open subset of

C = AUB.

For each x

and let V =

g >_ 1, E a subset of

ip is of

Y - U.

$= V^{-1}(V)$
choose an arc y_v
Baire class
Set
at x such
lim z->x f(z) $^{z €Y}$x
^{Y}x
^{Y}x
|z - x| £ 1}
- ’ {x} c f^{-1}(V)
Notice that if x G A and
We will say that
have subarcs y ’ and y ’ ’x ’y
if
if
x € B.
y G B, then
y„ meets y.
’ X
in
respectively such
n’ $^{311(1\ Y}$x’ $^{n\ Y}$y* *
L = {x€ A : (Vn)(3y)(y € C, a
M = {x € a
= {x’€
Let
and
: (Vn).(3y)(y€ C
: (3n)(v meets no x
: (9 n)(y meets no
L ^ULh
M = M_a U .
and
y ” *●
provided that y and y x y
that x £ y_x’ C H_n,
meets y in H)} y ’ x n’
meets y
in$_{Hn}$)}
Y_y (with y | x) in H_n)}
Y_y (with y f x) in HJ

Observe that $L_{\&}$, 1^, $M_{\&}$, are pairwise disjoint, and that and B = L^ u M&.
For
meets no y y
meets no y y
each x6M, let n(x) be a positive integer such that *y*
(with y
x) in . Then n >_n(x) implies that y_x
Let
• meets X_n, and, if x € M, n > n(x)}.
Then K K . for each n, and C = (7 X . n n+1 ' n
n=l
We next show that for each positive integer n and each x there exists a nondegenerate closed interval I® such that
x € C L_a (X - K^) . By the definition
y € C (y 1 x) such that y meets y_x in
interval having its endpoints at
x and
y-
of L , there exists a'
Let I^n be the closed x
Let t be any point of
We must prove that t £ $L_{\&}$ u (X - assume t £ K . Then y. meets X n [1]1 n
that y. must meet either y or y 't 'x '•
K) . n'
If t i K .we are done. ~ n'
and hence it is clear from
rigorized by means of
Theorem 11.8
in (This argument can ; on p. 119 in [11].) But,
then (because t € K_n)
n >_ n(t), so
Therefore
t £ M. Now
Hence y
C - B = A.
' I^n. x
So
Figure 5
be
if t € M,
that this situation^is impossible.
x £ L £ A, so, since y intersects y , a • ' 'x 'y*

Similarly, since y intersects y or y
☒x 'y

tec
= A.
Thus t £ A - M = L , and we have shown that a

(X -
K).
Let W n
x€ L
For each n,
a
L c W n C *Q* [L u (X - K)] A c, Cl 11 d 11
and therefore

Vac n=l co
☒ {p| [L u (X - K_n)]} n c '
n=l co
- I^{L}_{a} u (X -C[K n)] nC
CO.
= (L a C) u (C - U K) = L u <j>= L . .
d <t 11 CL d
n=l
co
It follows that L = (fA W_n) a C. By Lemina 19, each is equivalent n=l .
to an open set, so there exists a ($. set G_a *Q* X such that
I
L - G a c. a a
• • ☒
A similar argument shows that there exists a ($ set G^ ?= X such that
L_b " %AC-
Next we study the properties of M . In doing this, it is 81 '
2 convenient to define a function tt : R ☒* R by setting ir(x, y) = x.
If x € M nK , then, starting at x and proceeding along y , let p (x) n * • • x I*
be the first point of X that is reached. Define h° : M AK •* R r n n n
by setting h°(x) = ir(p_n(x)). If x, x' € M a and x < x', then, since
y cannot meet y ' in H , it is evident that p fx) must lie to the 'x x n n
left of p_n(x'); that is, ir(p_n(x)) < Tr(p_n(x')). Thus h° is a strictly
increasing function on M A K^. Moreover, •
|x - h°(x) | £ 1 because y_x C {z : |z - x| <_ 1}.

So, by Lemma 20, h° can be extended to a weakly increasing function
h : X -> R. Let n
g_nCx) = ,f(h_n(x),l) (x € R).
f(x, is a function (of x) of Baire class £(X, Y), so, by Lemma 21,
there exists a countable set N X such that g $|_v$ is of Baire class n • n^1X-N
— n
$N_{n>}$ Then is of Baire class
n=l
€(M - N, Y) .
For x € M A K_n, g_n(x) = f(h°(x), £) = f(p_n(x)). If x € M, then
for all sufficiently large n, x € M AK_n, so —
$_n$^> f(P_nW) = V>(X).
(pl., hence «pL, $_{XT}$ is of Baire class $_n$ t im-N' ^{T} 'M-N
Thus g I.. ⊠ ^{6}n'M
' ? + 1(M -
A
Obviously
L
^{so} sJm-n
N, Y). It follows that there exists D G $P^{^+2}$(X) such that a(m-n) = $(vl_{M\_N})^{-1}$(u)
= DA(M-N).
A a M ~ DAM. Now,
= L V L " fG> A Cl U fG. A Cl = fG v G, 1 A C
a n 'a ' ' d ' 'a d'
so
I
M_a = *A n* M⊠ - P n M = D A (C - L)
= D A [C - ((G_aU GJ,) A C)]
= -D-n [x - (G_a u g^] a c.
G_a and G^ are G$, so X - (G_a u G^) is , and hence
^{x} - (G_auG_b)€ P^2(X) Cp$£^{+2}$$_{(X)}$.
Therefore $M_{\&}$ « F A C, where F € $P^{?+2}$(X). Now, G_a € G_g(X) = Q^2(X), and since
E, > 1, Q^2(X) £ P^{5+2}(X), so G u FGP^{C+2}(X). But a
A = L u M - (G A C) U (F A C) = (G_ v F) A C, *AAA* d
so A - S a 0, where S € $P^{^+2}$(X). Since every countable set is F , it o
is now easy to show that
A = T n C
for some T € P^+^(X). From the definition of C it follows that
T — X - B. Thus we have
= ACTHECE-B = E - cp^{-1}(V) =
THE *G* P^+2(E), so Lemma 22 shows that is of Baire class g + 1(E, Y) . ⊠

Corollary. Let Y be a separable arcwise-connected metric space, f : H -> Y a Borel-measurable function, E a subset of X, and <p: E ->Y a boundary function for f. Then vp is Borel-measurable.

Proof, f is of some Baire class 5CH, Y), hence <p is of Baire class g + 1(E, Y), hence (p is Borel-measurable. ⊠

This corollary raises the question of whether a boundary function for a Lebesgue-measurable function is necessarily Lebesgue- measurable, which we answer in the next section.

7. Boundary Functions for Lebesgue-Measurable Functions

i

Suppose that a_Q, b , a^, b^ are extended real numbers, and that a < b , a; < b.. To make the formalism more convenient we let 0—0* 1—1

(-«>)- (-») = 0 and (+<») - (+00) = 0. In other respects we adhere to the usual conventions regarding arithmetic operations that involve -co or +00. Let

T(-a0' bo' al» bP = < : 0 £ y £ 1 and

(al - ao)y + a_o £ x £ (b_x - b_Q)y + b_Q}.

A set of this form will be called a *closed trapezoid*. We also consider <|>to be a closed trapezoid. A set S will be called a *trapezoid* if there exists a closed trapezoid T such that T^1 *Q* S ST, where T^1 denotes the interior of T relative to H^. Every trapezoid is Lebesgue-measurable, though not necessarily Borel-measurable. *

If s, s' are disjoint line segments having endpoints ^a_Q, 0^ <a_1, 1> , and <a_Q', 0> , <a_1', 1> respectively, where a^ £a^' (i = 0, 1), then let

T(s, s') = T(s', s) = T(a_o,,a_o'; a_p a_J').

If s = s', then we let_T(s, s') = T(s', s) = s. In what follows we will use the symbol X_Q as an alternative designation for the x-axis X. This will enable us to make statements about X^ (1=0, 1) (where Xj, denotes, as before, { <x, O : x€ R}).

We omit the proofs of the following two routine lemmas.

Lemma 23. Let theline segments s', s_, s~, s. each have one endpoint X Z *O*

on X_Q and the other on X^, and assume that i 4 j implies that either

s. n s. = $ or s. = s.. If T(s,, s_9) n T(s,, s.) 4 <l»then

1 J .1 J AZ ⊠ 4

T($s_{1?}$ s_3) *Q* T^, s_2) u T(s_3, s_4).

Lemma 24. Letbe any set of line segments, each of which has one endpoint on X_q and the other on X^, and no two of which intersect.

Then T(s, s') is a trapezoid.

s,s'€£ 2

Let m denote two-dimensional Lebesgue measure in R . If E

2 *Z*

is a measurable subset of some line in R , let m (E) denote the linear

—— £

Lebesgue measure of E. Let m_g and m_g denote two-dimensional exterior measure and linear exterior measure, respectively; i.e., for any E $c_R{}^2$;

m (E) = inf {m(U) : E U and U is open); 6

67

t

and if E is.a subset of a line L, then p' p _

m*(E) = inf {m (U) : E L and U is open relative to L).

Theorem 9. Let «C be any set of line segments, each of which has one endpoint on X_q and the other on Xp and no two of which intersect. Let S = . Then

i m (bj = | (mUS n X) + mHS n X)) . C 4 U U C X

Proof." We may assume that <£ is nonempty. Let e be any positive

2 number. Choose an open set U 5= R such that S <=. U and

m(U)' £ m (S) + e.

Let E^ = S n X^ (i = 0, 1). Choose sets <=. X^ that are open relative to X. such that E. *Q.* G. and i ii

p p

m (G.) < m\E.) + e (i = 0, 1).

1 " C X

2

Let V be the union of all lines L Cr such that L meets both G and o

Gp It is easy to show that V is an open set. Furthermore, S £V and V n X. = G. (i = 0, 1). Now let W = U n Y. Then i i *• '

W is open, S C W S U, and -

E. C W n X. C G. (i = 0, 1).

If s, s'€ <£ , define s = s' if and only if T(s, s') *Q* W.

It is easy to verify by means of Lemma 23 that = is an equivalence relation. Let T be the set of all equivalence classes. We prove that r is countable.

If s € , we let ^a^(s) , i^> be the endpoint of s on X^

(i = 0, 1). Then ,

2

s = { (x, y^^R : 0 _< y 1 and x = (a^s) - $a_Q(s)$)y + $a_Q(s)$}.

Since s is compact and contained in W, there is no difficulty in showing that there exists V > O .such that

£ {x, y> € R : 0 £y < 1 and

(a, (s) - a (s))y + a (s) - & . x < ($a_1(s)$ - a (s))y + a (s) + 6 } XU U 3 X v v O

Q w.

Let <K(s) = (a^(s) - $\&_s$, a^(s) + S_g) (i = 0, 1). A sketch will rapidly convince the reader that if s, s' €«C , $J_Q(s)$ n J_0(s') 4 and $J_x(s)$ n J^(s') 4 'I', then T(s, s') *Q* W, so that s = s'. Thus

($J_o(s)$ x $j_{i(s)}$)) A $CJ_0(s')$ x $J_x(s')$) 4 <> =*>. s' = s'.

For each C € F, choose s(C) € C and let

Q(C) = J_0(s(C)) x J^sfC)).

Then C. 4 C_9 =^Q(C.) n Q(C_9) = $. Since each Q(C) is a nonempty 2

open subset of R , this implies that F is countable.

If C6F, let

T(C) = VJ T(s, s’), s, s’GC

By Lemma 24, T(C) is a trapezoid. Also,

(35) CCt(C)$W.

Suppose that C^, C_2 £F and 4 ^{C}2’ ^e claim that

T(C..) T(C_9) = <j>. Assume that T(C.)nT(C_9) 4 • Then there exist

S„ s_n ‘ € C. and s_9, s_9’ £ C_9 such that T(s_n, s’) A T(s_9, s_9’) 4 4» •

By Lemma 22,

T(s_1, s_2) C T(s_1, Sj’) u T(s_2, s_2’) *C* W,

so that s. = s_9; a contradiction. Therefore T(C.) A T(C_9) = <|>. X I X £»

Let . IL (C) = T(C) a $X_{\pm}$ (i.= 0, 1). . Then IL (C) is an interval and

(36) E. C K. (C) C.W AX. CG (i = 0, 1).

¹ cer

Furthermore, C. C_o implies that K.(C.) A K.(CL) = $. Using the X Z XX X £»

formula for the area of a trapezoid, we find that

’ 4 [A U K (O) + im^C U K (C))]

= cgr C€r

= s’ 4 (m*(K (O) + $m^£$(K (C))) cer ^{z}

= *E m(T(C))* = m(U T(C)). cer cer

Let a = 4 [m\U $^{+ K}$iCC))] ^{z} cer cer

= iucI^Jtcc)).

cgr

According to (35), S *Q \|)* T(C) *Q* W *Q* U, so that cer

(37) m (S) < a < m(U). < in (S) + e. e — — c

By (36),

(38) | (m*(E_o) + ^(Ep) < a < | ($m^£$(G_Q) + $m^£$(G_1))

<7 CmhE) + m*(E)) + $_e$.

Since e is arbitrary, inequalities (37) and (38) imply that

m (S) = | (m*(E) + m*(E)). ⊠ :

One wonders to what extent a result resembling the foregoing theorem might be obtainable without the hypothesis that no two of the line segments intersect. The’following example is relevant to this question. Let M_q be a residual set of measure zero in/X_Q and let be a.residual set of measure zero in Let <x_q, y\ be.any point of Hj. We claim that there is . a line segment passing through ^x_Q, *y*^ that has one endpoint in and the other in M^. For 0 € (0, it), let

Fife) = <(1 - y_0) ctn 6 + x_o, 1> and

f_0(q) = <$^x{}_0$ - y_0 $^{ctn\ e}$> ^{0} >•

Then F^ is a homeomorphism of (0, it) onto X^, so $F_q^1(M_o)$ and F^^(M^) are both, residual sets in (0, it). Choose a £ ^{F}o^M_o) HF^^(M^). Let L be the line whose equation is

$x = x_q + (y - y_o)$ ctn a.

Then L passes through the points ^x_q, *yj)* , $F_Q(a)$ and $F_1(a)$, so that LaH^ is the desired line segment. Let be the set of all line segments having one endpoint in M_q and the other in M^. Let S = Then S *r*\ X_q and S *f*\ X^ both have measure zero, but, as we have just

1 - '

shown, S, so that S has infinite measure. See Problem 5 at the end of this paper.

Lemma 25.

For every e > 0 there exists a strictly increasing real-

valued function h on R such that h(R) has

measure zero, and, for every

real x, |x - h(x)| £ e.

I +" I

Proof. For each integer n, let I = [ne, (n + 1)e]. Then I_n = R« n^ioo

There exists a strictly increasing function f : [0, 1] •* [0, 1] such that m\f([0,1]))= 0. For example, such a function may be defined as follows. Any number in [0, 1) may be written in *ti&* form

•$,a9a,.. .a. ... (binary decimal),

A *bi O IX*

where the decimal does not end in an infinite unbroken string of 1's.

Set.

f (-a..a,,a,.. .a ...) = •b-.b-b .. .b ... (ternary decimal), ⊠ x z n ' x z o n

where b. = 0 if a. =0 and b. = 2 if a. =1. i i i i e

Set f(l) = 1. Then f maps [0, .1] into the Cantor ternary set, so

£

m (f([0, 1])) = 0. It is easily shown that f is strictly increasing.

For each n, it is easy to obtain from f a function f : I ⊠* I 9 J n n n

£

such that f is strictly increasing and m $(f_n(I_n))$ = 0. Set

h(x) = f^fx) for x C(ne> Cn + 1)£]-

There is no difficulty in proving that h has the required properties.®

Theorem 10. There exists an indexed family £y_v}v Y simple arcs X X C A

such that

(i) for each x g X, y is an arc at x X

(ii) x 4 y =^y_v ny_vM

(iii) I *J* y is a set of measure zero, x € X X

Proof. For each natural mumber n, let h_n : R •* R be a strictly increasing function such that $h_n(R)$ has measure zero and, for every x, |x - h_nW | For every x € R, let $s_n(x)$ be the line segment joining the point ^$h_n(x)$, to the point ^h_{n+}^(x), Since

< hntx2] >^x1 < ^X2^hn+1(X1) < hn+1tx2)'

we see that x^ implies sn(x^) n Sn^) = <|>. Let Sn = VJtsn(x) : x£ R}. Then

S n X C{(x, : x€ h (R)}

n n — ' ' n ' nk '

and S n X . C. {/x, -A-*5 : x g h n (R)}, n n+1 - x » n+1 / n+1k J *

g £

so m'(S n X) = m (S n X .) = 0. It is easy to deduce from n n n n+1

Theorem 9 that ;

1'1 If 2

VSn> • < K - nJT ' T CVSn " V * VSn n Xn+1» ' '•

For x € X, let y = {x} U I) s (x). Since \h (x), — / •* x, y is x n n n / n x

an arc at x.

00

Ⓡe(U Yx) < meW + me(U SJ

e x£X x e e n=l n . co

£m (X) + E m (S) = 0, e n=l e n

so y is a set of measure zero. ⊠ x6X x

Corollary. Let <p be an arbitrary function mapping X into any topologi

cal space Y having an element called 0. Then there exists a function

f : H -> Y such that f(z) = 0 almost everywhere and ipis a boundary

function for f.

Proof. If {y } YthⓇ family of arcs described in Theorem 10, let X X £ a f(z) =0 if z is in no y

X

f(z) = <f>(x) if z €Y . A

Then f is the desired function.Ⓡ

Corollary. There exists a real-valued Lebesgue-measurable function f defined in H having a nonmeasurable boundary function defined on X.

SOME UNSOLVED PROBLEMS

1. If A is an arbitrary set of type F_{a5} in X, does there necessarily exist a *real-valued* continuous function f defined in H having A as its set of curvilinear convergence? If <p is an arbitrary real-valued function of honorary Baire class 2 on A does there exist a continuous real-valued function f defined in H having A as its set of curvilinear convergence and (p as a boundary function?

2. (First proposed by J. E. McMillan [10]). If A is any set of type 2

F_ag in X and if (pis any function of honorary Baire class 2(A, S), 2

does there necessarily exist a continuous function f : H -> S having A as its set of curvilinear convergence and <p as a boundary function?

3. If f is a real-valued Borel-measurable function defined in H, is the set of curvilinear convergence of f necessarily a Borel set? What if f is assumed to be of Baire class 1?

• 3 -

4. Let S = { <x, y, z) g R : Z >0}. If f is a function defined in S, we define the set of curvilinear convergence of f in the obvious way. If f is continuous, is its set of curvilinear convergence necessarily a Borel set? Is it necessarily of type F ?

5. Let <£ be a set of line segments each having one endpoint on X_q and the other on X^, and let SUz . Assume that S is a Borel set.

£ £

If m (S n X_Q) and m (S n X^) are known, what lower bound can be given for m(S)? . A solution to this problem might be helpful in attacking a problem of Bagemihl, Piranian, and Young [3, Problem!].

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NOTE ON A PROBLEM OF ALAN SUTCLIFFE

T. J. KACZYNSKI, The University of Michigan

If n is an integer greater than 1 and *ah,* • • , ab a**o** are nonnegative integers, let

(ah, • • • , ai, ao)n denote ahnh + • • • + ain + ao.

Thus if O;;;ai;;;n-1 (i = O, • • •, h), then *ah, ^^,* a1, a**o** are the digits of the number (ah, • • •, ab ao)n relative to the radix n. Alan Sutcliffe studied the prob- ,

lem of finding numbers that are multiplied by an integer when their digits are reversed *(Integers that are multiplied when their digits are reversed,* this Magazine, ?

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Similarly,

(3) 6* = - 1.

Taking q = characteristic of F (g-l=O), choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

(Z + *ra* + i)(r» + 1 + *ria* + *lb)* = r(Z2 + r* + l)a + (Z2 + r2 + 1)6 = 0.

One of the factors on the left must be 0, so for some numbers *u, v, w, u* 0 (mod *q),* we have *w+va-\-ub* = *Q,* or *b* = — *u~lva* — *u~lw.* So *b* commutes with *a,* a contradiction. We conclude that S is not a generalized quaternion group, so 5 is cyclic.

Thus every Sylow subgroup of F^* is cyclic, and F^* is solvable ([1], pp. 181— 182). Let Z be the center of F* and accnme ⁷F^*. Then F^*/Z is solvable, and its Sylow subgroups are cyclic. Let A/Z ("fit.. ZC^) be a minimal normal subgroup of F^*/Z. A/Z is an elementary abelian group of order p^k *(p* prime), so since the Sylow subgroups of F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be qny element of F^*, y any element of A. Since A is normal, *xyx~'(E.A,* and (1 *+x)y* = z(l +x) for some z£Z. An easy manipulation shows that *y — z = zx — xy = (z — xyx~')x.*

[1] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,*

If y — z = z — $xyx^{\sim'}$ 0, then y = z = xyx~^l, so x and y commute. Otherwise, x (z — $xyx^{\sim 1})^{\sim'}$(y — z). But A is abelian, and z, *y, xyxr'^A,* so x commutes with *y.* Thus we have proven that A is contained in the center of F*. a contradiction.

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y e A(n + 1, m[n+ 1], k[n + 1], f [n 4- 1]) C U_{n+1} c Q(n 4-1, m[n+ 1]),

2tt i 1

and therefore each point of y_n has distance less than from y. Now

^+~F 0 ^{a}s n -* °°; hence, if we set y = {y} U $U_{n=}$i y_n, then *y* is an arc with one endpoint at y.

Since U_n and U_{n+1} have a point in common,

$^{f-1}$(* ^{S}(^> ^{p}k[n])) $^{and\ rl}$(^{S}(^Tl' Pk[n+1]))

have a common point, and hence

^{S}(^>Pk[n]') $^{and\ S}$(^+l' Pk[n+1])

have a common point. Therefore, if p is the metric on K, then

< \ ^{1} . ^{1} / *1* , ^Pk[np Pk[n+1] — 2^n $2^{n"\wedge}$

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Explanation by John D. Bullough

A function p(e) defined on the unit circle is a boundary function for a function f(z) defined in the unit disk provided for each e, f(z) has the limit p(e) at e along some curve lying in the unit disk and having one endpoint at e. Any two boundary functions for the same function f differ at only countably many points by the ambiguous-point theorem of Bagemihl; and a boundary function for a continuous function differs from some function in the first Baire class at only countably many points. In answer to a question of Bagemihl and Piranian, the author constructs a bounded harmonic function having a boundary function that is not in the first Baire class. He shows that nevertheless the set of points of discontinuity of such a boundary function is a set of the first Baire category.

Article by Ted

BOUNDARY FUNCTIONS AND SETS OF CURVILINEAR CONVERGENCE FOR CONTINUOUS FUNCTIONS

BY

T. J. KACZYNSKI

Let D be the open unit disk in the complex plane, and let C be its boundary, the unit circle. If x e C, then by an *arc at x* we mean a simple arc y with one end point at

x such that *y-{x}^D.* If /is a function mapping D into some metric space *M,* then the *set of curvilinear convergence* of *f* is defined to be

{x e C: there exists an arc y at x and there exists a point *pEM* such that /(z) -> p as z x along *y}.*

If is a function whose domain is a subset E of the set of curvilinear convergence of /, then</> is called a *boundary function* for / if, and only if, for each x *e E* there exists an arc y at x such that /(z) -></> (x) as z -> x along y. Let S be another metric space. We shall say that a function</> is of *Baire class* 1(S, *M)* if

(i) domain ^ = S,

(ii)range and

(iii) there exists a sequence of continuous functions, each mapping S' into *M,* such that <*f*>$_n$ -> <£ pointwise on S.

We shall say that is of *honorary Baire class* g 2(S, M) if

(i) domain <£ = S,

(ii)range</> ^Af, and

(iii) there exists a countable set *N^S* and there exists a function 0 of Baire class 1(5, Af) such that <£(x) = ^(x) for every *xeS-N.*

It is known that if/is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of / is of type F_{a69} and any boundary function for/is of honorary Baire class ^2(C, Riemann sphere). (See[1],[2],[3],[4], [9].) J. E. McMillan[5] posed the following problem. If A is a given set in C of type F_{ad}, and if is a function of honorary Baire class *^2(A,* Riemann sphere), does there always exist a continuous function / mapping D into the Riemann sphere such that A is the set of curvilinear convergence of / and is a boundary function for /? The purpose of this paper is to give an affirmative answer to McMillan's question. However, the corresponding question for real-valued functions remains open. (See Problems 1 and 2 at the end of this paper.) In proving our result, we first give a proof under the assumption that</> is a bounded complexvalued function, and we then use a certain device to transfer the theorem to the Riemann sphere. As we shall indicate in an appendix, the same device can be

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used to transfer certain results concerning real-valued functions of the first Baire class to the case of functions taking values on the Riemann sphere.

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Our proof is divided into several major steps, which are labeled (A), (B), (C), etc. The proofs of some of the major steps are divided into smaller steps, which are labeled (I), (II), (III), etc. The results (A) and (B) are taken from the author's doctoral dissertation[6].

Throughout this paper we shall use the following notation. R denotes the set of real numbers, S^2 denotes the Riemann sphere, and R^n denotes w-dimensional Euclidean space. Points in R^n will be written in the form <x_b x_2,..x_n> (rather than (x_{15} x_2,..., x_n)) in order to avoid confusion with open intervals of real numbers in the case n = 2. The empty set will be denoted by 0. When we speak of a complex-valued function, we mean a function taking only *finite* complex values. The closure of a set E will be denoted either by E or by Cl E. If I is an interval of real numbers, then Z* denotes the interior of I. If p is a point of some metric space and r e (0, 4-oo), then $S(r, p)$ denotes the set of all points of the space having distance (strictly) less than r from p.

We define

Q = {(x,y)eR^2 : -1 x 1,0 < y 1}, X = { <x,0>: -1 < x < 1}, H = { <x, y>e R^2 : y > 0}.

It will be convenient to identify <x, 0>with the real number x, and X with (—1, 1). If f is a complex-valued function defined in Q, then we shall understand the set of curvilinear convergence of f to mean the set of all x e X for which there exists an arc y at x (contained in the interior of Q except for its end point at x) such that f approaches a finite limit along y. If a e X, e > 0, and 0 < 0 < then we let

$s(a$, e, 0) = {<x, y) e R^2 : 0 < y < e, a—y ctn 0 < x < a+y ctn 0}.

Thus $s(a$, e, &) is the interior of an isosceles triangle in H with apex at a.

(A) *If A ^X is a set of type* F_{a6i} *then there exists a bounded continuous realvalued function g defined in Q such that*

(i) *for each xe A, g(z) ->* 0 *as z approaches x through* $s(x_9$ 1, |tt), *and*

(ii) *if xe X, and if there exists an arc y at x such that g(z) ->* 0 *as z approaches x along y, then xe A.*

(I) Let $E_{\pm}$ and E_2 be two sets on the real line. A point pE R will be called a *splitting point* for E_x and E_2 if either

(i) Xi ^p for all $x_{\pm}$ e E_y and p x_2 for all x_2 e £$_2$, or

(ii) x_2^p for all x_2 e E_2 and p^x_1 for all x_r e $E_{\pm}$.

We will say that $E_{\pm}$ and E_2 *split* if and only if there exists a splitting point for E_t and E_2.

(II) By a *special family* we mean a family of subsets of X such that

(i) & is nonempty,

(ii) & is countable,

(iii)each member of & is compact,

(iv)if E, Ftthen either E=F, Ec\ F=0, or E and F split.

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[6] F. Hausdorff, Uber halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

(III) If £⊠£ JTis an F_a set, then there exists a special family & such that E= (J

Proof. We can write E= (Jn= i where Ai = 0, A_n is closed, and ^4ns^n+1 for all *n.* Observe that if I is any open interval contained in X, then there exists a countable family {Jn}"= i of compact intervals contained in X such that *1*= IJ "= i A, and n/m implies that J_n and J_m split. Since X-A_n is a countable disjoint union of open intervals, it follows that we can choose (for each ri) a family

i

of compact intervals such that X— A_n— U"i and j£k implies that I_{nJ} and Zn,fc split. Let

= *{Ai}* u $\{I_{nJ}$ n ^tt+1 : n = 1, 2,...; j = 1, 2,...}.

Then *SF* is a countable family of compact sets, and

E = u A = Ai V U A+1 n (X—An) n=1 n=1

= Ai u (J (J ^4n+1 ri Inj n=l 1=1

=

Let *Fi* and F_2 be any two distinct members of If either F_x or F_2 is A_x = 0, then *Fi* and F_2 are automatically disjoint. If neither F_r nor F_2 is A, then we can write

Ei = 4i(i),«i> 24n(i)+i,

E_2 — fn<2)J(2) A_{nm+1}.

If n(l)<n(2), then n(l)+1 ^n(2), so

Ez *fn<2),/(2) ^n(2) + l £* X— *^4n(2)* — X— *^n(l) + l* — X— Fi,

and therefore $F_\pm$ and F_2 are disjoint. If n(2)<«(l), a similar argument shows that *Fi* and F_2 are disjoint. Now suppose n(l)=n(2). Then, since Fi/F2, we have j(l)/j(2). So /n(i).«2)=A(2),/(2) and Lxd.xi) split, and consequently F_r and F_2 split. We have shown that any two distinct members of & either split or are disjoint, so

is a special family.

(IV) Let A £ X be a set of type $F_{a!>}$. Then there exists a sequence of special families {^}"=i such that

(0 ^An^JU^n),

(ii) if 1 and *Ee*^+1, then there exists *F*with *E^F.*

Proof. There exist sets *Ai* 2^22^32 • • • such that A *A"=i* A_n. *By (III), we can choose (for each n) a special family* A_n *such that* <em>A_n (J <^n- Let For 1, let

•^+1 = {fn *E :* and £e«fn+1}.

By induction on *n*, one can show that each is a special family and that A_n = (J It is clear that the other conditions are satisfied.

(V) Suppose that J is a nonempty interval with *X,* and let a, *b (a b)* be the end points of *J.* By Trap (J, e_9 *0)* (where *0 e* (0, -^*r)* and *e* > 0) we mean the trapezoidshaped open set defined by

Trap (J, e, *0)* = {<x, : 0 < *y* < *e, a*+*y* ctn *0* < *x* < *b*-*y* ctn 0}.

For *0 e* (0, i^) let Tri *(J,* 0) be the closed triangular area defined by

Tri *(J, 0)* = {<x, *: y* 0, *a*+*y* ctn 0 x *b*—*y* ctn 0}.

If K is a nonempty compact subset of X, let $J(K)$ be the smallest closed interval containing K. If $e > 0$ and 0</?<a<|77, then we define

$B(K_9$ e_9 $a,$ $0)$ = Trap (/(£), e, a) - U Tri (Z, $0)_9$

leS

where</ denotes the (possibly empty) set of disjoint nonempty open intervals whose union is $J(K)$—K.

We state without proof the following readily verifiable facts ((VI) through (XVIII)).

(VI) $s(x_9$ e_9 0) is an open subset of H.

(VII) Cl [s(x, s, 0)] n y={x}.

(VIII) If e<e and 0'<0, then Cl [s(x, e, 0)] n H^$s(x_9$ e_9 0').

(IX) If x/j and c, 0 are given, then there exists 8>0 such that, for every 77 8, j(x, e, 0) and s(y_9 t?, 0) are disjoint.

(X) $B(K_9$ e_9 a, $0)$ is an open subset of H.

(XI) If $K_\pm$ and K_2 split, then, for any e_2, a, and 0_9 $B(K_{19}$ s_{19} a, £) and Z?(/C_2, e2, $0)$ are disjoint.

(XII) If K+ and K_2 are disjoint compact subsets of X_9 and if e_9 a, 0 are given, then there exists 8 > 0 such that for every 77 8, $B(K_{19}$ e_9 a, $0)$ and $B(K_{29}$ 77, a, $0)$ are disjoint.

(XIII) Cl $[B(K_9$ e, a, $0)]$ n X^K.

(XIV) Suppose that K^K, c>$£_1$>0, and 0<j3<^{1}<a_1<a<7r/2. Then Cl [B(^, 8_{19} a_{19} $0,)]$ n H^$B(K_9$ $8,$ a, $0)$.

(XV) Suppose that a<0<fyr and x^J(X)*. Then, for any e_9 e_{19} and 0_9 $B(K,$ $e,$ a, $0)$ and $s(x_9$ e_{19} 0) are disjoint.

(XVI) Suppose that $x\$K$ and that e, a, 0_9 0 are given. Then there exists 8>0 such that for every 8, $s(x_9$ 77, $0)$ and $B(K_9$ e_9 a, $0)$ are disjoint.

(XVII) Suppose that $x\$$ K and that 8, a, 0_9 0 are given. Then there exists $s > 0$ such that for every $s(x_9$ 8, 0) and $B(K,$ $^_9$ a, $0)$ are disjoint.

(XVIII) Suppose that xe $Kc\backslash J(K)$* and O<0<a<0<|7r. Let e be given. Then there exists 8>0 such that for every 77 8, Cl [s(x, rj_9 0)] n $B(K_9$ e_9 a, $0)$.

(XIX) If & is a special family, let «^{2} be the set of all members of & that have two or more points, and let E(^) be the set of all end points of intervals $J(F)_9$ where F e $\mathscr{E}$ and f / 0.

Suppose that O</3<a<0<|w, and that SF is a special family. By a *pair of special a, fl, 0 functions for* I mean a pair («, 8), where e and 8 are positive real-valued functions, the domain of e is $E(\mathscr{E})$, the domain of 8 is &^{2}, and

(i) for each > 0, there exist at most finitely many F e $\mathscr{E}^2$ such that 8(F) -q;

(ii) for each $q > 0$, there exist at most finitely many x e $E(\mathscr{E})$ such that e(x) $t]$;

(iii) if $x,$ x' e $E(^)$ and x#x', then $s(x,$ «(x), $0)$ and $s(x',$ «(x'), $0)$ are disjoint;

(iv) if $F,$ $Ke\mathscr{E}^2$ and F*K, then $B(F,$ $8(F),$ $a,$ $fl)$ and $B(K,$ $8(K),$ $a,$ $fl)$ are disjoint;

(v) if x e $E(\mathscr{E})$ and FeF^2, then $s(x,$ e(x), $0)$ and $B(F,$ $8(F),$ $a,$ $fl)$ are disjoint.

(XX) Let & be a special family and suppose that O</3<a<0<|?r. Then there exists a pair of special $a,$ $fl,$ 0 functions for

Though a formal proof of this statement is lengthy, it requires no originality, so we omit the details. The idea is to arrange the members of & in a finite or infinite sequence F_x, F_2, F_s,..., and then define *e* and 8 inductively. One makes use of statements (IX), (XI), (XII), (XV), (XVI), (XVII).

(XXI) Let & be a special family, and suppose O</?<a<0<£n-. Let («, 8) be a pair of special *a, fl, 0* functions for If $e_{1(}$ 8j are two real-valued functions having domains *E(&)* and $\&^2$ repectively, and if

0 < ej(x) g «(x) for all x e *E(&'),*

0 < $8_X(F)$ *8(F)* for all $Fe\&^2$,

then («!, 8J is a pair of special a, *fl, 0* functions for *3F*

The proof of this statement is trivial.

(XXII) We now proceed to the proof of statement (A) itself. Let *A* be our given F_{06} set. By (IV), we can choose a sequence of special families such that *A* = (1"₌i (U ^*n),* and for each Ke_{+1} there exists *Fe* with *K^F.*

Let {j8$_n$}"=i be a strictly increasing sequence in (0, jw) coverging to |?r.

Let {a$_n$)„=i be a strictly decreasing sequence in (in-, Jw) converging to *in.*

Let {^$_n$}n=i be a strictly increasing sequence in *(in,* jw) converging to

Let E_n=*E(*^$_n$*).*

Let (e(l, •), 8(1, •)) be any pair of special aj, fl_{lf} 0_Y functions for

Now suppose that for each *k^n* we have chosen a pair of special «k, *Pk, Qk* functions *(e(k, •), 8(k,* •)) for in such a way that

(i) whenever 1 ^*k^n* — 1, x e E_{k+1}, *FeF*$_k$, and x *e Fc*\ *J(F)*,* then

Cl [s(x, *e(k*+1, x), 0$_{k+}$i)] n *H £ B(F, 8(k, F), a$_k$, fl$_k$);*

(ii) whenever *l^k^n-1, xe* E_{k+1}, and x e E_k, then

Cl [s(x, *e(k*+1, x), fl$_{k+1}$)] *H* £ s(x, *e(k,* x), 0$_k$);

(iii) whenever KKn-1, Xe («^+i)^2, *Fe* (^)^2, and *K^F,* then

CI *[B(K, 8(k+l, K), a$_{k+1}$, fl$_{k+1}$)] B(F, 8(k, F),* a$_k$, fl$_k$).

Then we construct (s(n+l, •), 8(«+l, •)) as follows. Let (e, 8) be any pair of special a$_{n+1}$, £$_{n+1}$, 0_{n+1} functions for «^$_{+1}$. If *x e* E_{n+1} — E_{n9} then for some unique Fe(^,)^7, *xe F C*\ *J(F)*.* By (XVIII), we can choose f(x)>0 so that ??^f(x) implies

Cl [s(x, *rj$_9$ e$_{n+1}$)] B(F,* 8(n, F), a_{n9}

We set e(n+1, x) = min {c(x), f(x)}. On the other hand, if x *e* E_{n+1} n E_{n9} then we set e(n+1, x) = min {e(x), ^*e(n$_9$* x)}.

If *Ke* («^$_+$i)^2, then there exists a unique Fe(^$_n$)^2 with *K<^F.* Set

8(«+1, *K)* = min {8(F), |8(n, F)}.

By (XXI), (e(n+1, •), 8(n+1, •)) is a pair of special a$_{n+1}$, £$_{n+1}$, 0$_{n+1}$ functions for *&n $_{+1}$,* and, by (VIII) and (XIV), conditions (i), (ii), (iii) are still satisfied when *n* is replaced by « + l. Thus we can inductively construct a pair *(e(n$_9$* •), 8(n, •)) of special a$_n$, ft_{n9} 0_n functions for in such a way that (i), (ii), and (iii) are satisfied for every *n.*

Let

U_n = r u $s\{x_9$ $8\{n_9$ x), 0_n)i U r U $b(f_9$ 8(w, f), a_n, $p_n)]$. L$^{xe£}$n J LFe^n)^2 J

Then U_n is open. For fixed n_9 all the various sets $s(x_9$ $e(n_9$ x), 0_n) (x e F_n) and B(F, 8(n, F), a_n, 0_n) (Fe(^)^2) are open and pairwise disjoint, so that every component of U_n is contained in one of the sets $s(x_9$ $e(n_9$ x), 0_n) (x e F_n) or $B(F,$ $8(n_9F)_9a_{n9}p_n)$ $(Fe(^\wedge{}_n)^2)$. It therefore follows from (VII) and (XIII) that if W is any component of U_{n9} then

(1)

From conditions (i) and (ii) in the definition of a pair of special a, £, *0* functions, it follows that

Un n H = r U Cl [s(x, $e(n_9$ x), 0_n)] n 771 u [J Cl [B(F, 8(w, F), a_n, &)] n 771.

l*-G^n J Lre($^\wedge{}_n$)2 J

Consequently, conditions (i), (ii), (iii) in our inductive construction of *(e(n₉ •),* 8(«, •)) (together with the fact that x *e* F_{n+1} — E_n implies x *eF J(F)** for some *Fe* («^)^2) imply that U_{n+1} n U_n for every *n.*

By Urysohn's Lemma there exists a continuous function g_n: H -> [0, 1] such that g_n(z) = 1 for *z e H*— U_n and g_n(z)=0 for z e l/$_{n+1}$ n 77. Let

g& = 2 n=l

Then 0^g(z)^ 1, and the series converges uniformly, so g is continuous in *H.*

If z e 77- U_{n9} then z e 77- U_m for every *m* n_9 so that 1 =g_n(z)=g_{n+} i(z)=g_{n+2}(z) = • • •, and hence

Also, if z e t/$_{B+1}$, then *z e* U_u U_2, ⊠ ⊠U_{n+1}, so that 0=gi(z)=g_2(z) = • • • =$g_n(z)$, and

(3) g(z) 2 Ⓡ^{m} = Ⓡ^{n} (zel/$_{n+x}$).

m = n + l

Let x_0 e *A* be given. We must show that g(z) -> 0 as *z* approaches x_0 through s(x_0, L i$^{77"}$)- Take any natural number *n.* Since x_0 g (J^$_{+1}$, it follows that either x_0GF_{n+1} or else x_0 g *FnJ(F)** for some Fe(J_{n+1})^2. In the first case, set *rj* = *e(n* +1, x_0). In the second case, (XVIII) shows that we can choose *rj>0* small enough so that

$s(x_0,$ *7],* >77) c *B(F,* 8(h+ 1, F), a_{n+} i, $_+$

Suppose <x, *y>e* $s(x_0,$ 1, *far)* and *y<rj.* Then, in the first case,

<X, *y)* 6 $s(x_0,$ *7), >77)* *Q* s(x_0, *e(n+* 1, X_o), 0_{n+} l) $^\wedge{}_{n+1}$,

and, in the second case,

<x, *y)* g s(x_0, *7J,* <77) C *B(F,* 8(n +1, F), a_{n+1}, p_{n+1}) <= U_{n+1}.

Thus, referring to (3), we see that g(x, *y)* (|)^n whenever <x, *y)* *e* 5(x_0, 1, $77) and *y* <*t].* Therefore *g(z)* -> 0 as *z* approaches x_0 through $s(x_0,$ 1, $77).

Let Xi be a point of *X,* and assume there exists an arc *y* at x_Y such that *g(z)* -> 0 as *z* approaches $x_\pm$ along *y.* Then *y* has a subarc *y* with one end point at Xi such that By (2), /-{x^^^. Therefore, by (1), $_{X1}$ *e*

U «^. Since *n* is arbitrary,

*1 6 A (U ^n) = *A. n* = 1

Thus, by restricting *g* to *Q* we obtain the desired function.

(B) *Let A be a subset of X of type F_{ad}, and let be a bounded complex-valued function of honorary Baire class ^2(A,* F^2). *Then there exists a bounded continuous complex-valued function h defined in Q such that, for each x e A, there exists an arc y at x with y—{x}^s(x,* 1, $77) *and*

lim *h(z)* = <£(x).

(I) Let Z be a bounded open interval in *R,* and let/: 7-> 7? be a bounded, strictly increasing function. Then there exists a continuous, weakly increasing function /: R-^ R *such that* f*(f(x))=x* for every *x e I.* (This result is probably not new, but I do not know of a reference for it, so I am obliged to prove it here.)

Proof. Let *Z=f(I),* let c=inf Z, and let <7=supZ. Observe that *Zs(c, d),* and that /$^{-1}$: Z-> *I* is strictly increasing. I assert that for each *x* e (c, *d)*

(4) sup/$^-$ \ *(c,* x] n Z) = sup/$^-$ \ *(c,* x) n Z).

If x £ Z, the equation is trivial. Suppose x e Z. Then

c < *y* < /$^{-1}$(x) => *(f(y)* < *x* and *f(y) eZ),*

so that (c,/$^{-1}$(x))£/$^{-1}$((c, x) Ci Z). Hence

sup/$^{-1}$((c, *x)r\Z) f~\x)* = sup/$^{-1}$((c, x] n Z).

The opposite inequality is trivial, so (4) is established.

I also assert that for each x e (c, *d)*

(5) inf/-^{X}((x, *d) n* Z) = sup/- '((c, x] n Z).

Obviously,

inf/'XCx, *d)* n Z) sup/-^{x}((c, x] n Z).

Take any y>sup/$^{-1}$((c, x] nZ). If /(y)^x, then *f(y) e (c,* x] n Z, and so *y*^{e}*f~K{c,* x] H Z), a contradiction. Thus *f(y)>x* and *f(y)* e (x, *d)* n Z. Therefore *y* 6/$^{-1}$((x, *d)* r> Z), so that inf/$^{-1}$((x, *d) c\Z)^y.* In view of the choice of *y,* this implies that

inf/'^x, *d)* O Z) sup/-^{x}((c, x] n Z),

and (5) is established.

Define/* on *(c, d)* by

/☒(x) = sup/$^{-1}$((c, x] n Z) (x e *(c, d)).*

It is clear that /* is weakly increasing and that *f*(f(x))=x* for every xeZ. The continuity of/☒ can easily be deduced from the equations

sup/*((c, x))* = /(x), inf/*((x, *d))* = /☒(x),

which are established as follows:

sup/*((c, x)) = sup sup/$^{-1}$((c, y] n Z) ***c<y<x***

= supx) Cl Z)

= sup/$^{-1}$((c, x] n Z)

= /*(x),

inf/*((x, *d))* = inf sup/$^{-1}$((c, y] n Z) ***x<y<d***

= inf inf/-^{x}((y, J) nZ) ***x<y<d***

= sup/$^{-1}$((c, x] n Z) = /*(x).

We now extend/* to all of *R* by setting

/*(x) = inf*d))* if *x c,*

/*(x) = sup/*((c,</)) if *x* >*d,* and we are finished.

(II)Suppose that *M* is a metric space and that *u: M ->R* is a function having the following property. For every sequence $\{p_n\}$ of points of Af, every *pe M,* and every *y* e *R* u {-oo, +oo}, if p_n->*P* and u(p_n) -> y as n -> oo, then *y g R* and *u(p)=y.* Under this hypothesis, *u* is continuous.

Proof. Let $\{p_n\}$ be any sequence of points in *M* converging to a point *pE M.* We have only to show that *u(p^* -> u(p). But suppose u(p_n) t/(p). Then there exists a subsequence {u(/?$_{n(fc)}$)} and there exists *y e R\J {—as,* +oo} such that *y±* ^{u}*(p)* and w(/*n(k)) *y* as *k* -> oo. Since $p_{n(fc)}$ ->*p* as *k* -> oo, this contradicts our hypothesis.

(III)Let *A ^(*-1,1) be of type F_{ad9} and let be a complex-valued function of Baire class 1(?1, *R*^{2}*).* Then there exists a sequence $\{g_n\}$ of continuous functions, each mapping *R* into $R^2{}_9$ such that for each *x e A,* $g_n(x)$ *-> i/j(x)* as *n* -> oo.

Proof. This can be proved in a more general context, as shown in[8]. For a quick proof of the special case stated above, we can refer to a theorem of Bagemihl and McMillan .[1, Theorem 2], which tells us that there exist continuous real-valued functions and f_2 defined in *H* such that, for each *x e* A_9 has angular limit Re (0(x)) at *x* and f_2 has angular limit Im (^(x)) at x. For each x *e* R_9 set

gn(x) = fdx, + if$_2$*(x,* iy

\ n) \ n/

(IV)Now we proceed to the proof of statement (B). Let 0 be a function of Baire class *1(*A_9 K^2*)* and let £ be a (possibly empty) countable subset of *A* such that <£(x) = ^(x) for each x e *A —E.* Let *N* be an infinite countable set with *E^N^X.* Let w be a real-valued function defined on *N* such that *w(s)* > 0 for each *se N* and

2 M<s) < $2^{1/2}$ —1. *sgN*

For each *xe X=(*— 1, 1), let *N(x)={s*e*N:* — l<s<x}. Define *f* on (—1,1) by setting

f(x) = x+ 2 HO-

seN(x)

Then/is a bounded, strictly increasing function on (— 1, 1), and |/(x)—x| < $2^{1/2}$— 1. By (I), there exists a continuous, weakly increasing function *f*: R-+ R* such that *f*(f(*^{x}*))=*^{x} for each x e (-1, 1).

Let

H_o = {<x,y>$_{6j}$$R^2$:O<y

For fixed <x, *y)* e H_o,

$_{u\_f}$*Jx-(l-y)u*

\ y /

is a strictly increasing continuous function of *u* that approaches +oo as *u* -> +oo and — oo as *u* -> — oo. Consequently there exists precisely one number u(x, *y)* that satisfies the equation

(6) *<&.y)* - o.

[8] F. Hausdorff, Uber halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.,* 5 (1919) 292-309.

I assert that $u(x, y)$ is a continuous function on H_o. We show this by using (II). Suppose that <x, y>e H_o, u_0 e E u {-oo, +oo}, {<x_n, y_n>}£/f0, On, y_n> -> <x, y>, and $u(x_n, y_n)$ -> w_0- If u_0 = +°o, then, as n ->oo,

x_n-(l-y_n)u(x_n, y_n) ; x y_n ’

and so

U<X_n, A)-/*(Xn~(1~^XXn’K)) +«>,

\ zn /

which contradicts (6). So i/0^ +°o, and a similar argument shows that m_o0 — oo. Thus, by (6),

0= lim [u(x_n, K)-/*(Xn~° ~y”)u(x” n->co L \ yn /J

-

Consequently u_0 = u(x, y). By (II), u is continuous.

From (III), there exists a sequence {g_n} of continuous complex-valued functions defined on R such that $g_n(x)$ -> 0(x) as n -> oo for each x e A. For n 2, define

$h_0(x, y) = (yn(n+1) - n) g_n(u(x, y))$ + ((n +1)-$yn(n$ + l))g_{n+}j(u(x, y))

when 1/(72 4-1) ^y 1 /«. Then h_0 is continuous on H_o. Let {s_n}n=i be all the elements of N, where w/zn implies s_n ^s_m. Let

r_n = inf /(x), ***X*>*S*$_n$** *In* = SUP /(X) = $f(s_n)$, ***x* < *s*$_n$**

Zn = if $s_n eE$,

z„ = 0 if s_n *$ E.*

Notice that r_n—/$_n$>0. If x and y are real numbers, define xvy=max{x, y} and x Ay=min {x, y}. For <x, y>6 H_o, set

A„(x,y) = [(1 -ny) v O]I(1

r_n+l_n-2s_n+2

v o z_n.

L \ ’n h

Then A_n is continuous in H_o. Observe that A_n(x, y)=0 when y^l/n. Using this fact, it is easy to show that, if we set

oo

hl = h_0+ 2 ⊠n, n = 1

then is defined and continuous on H_o.

Let p be any point of A. The line (7) passes through <p, 0>, and, since |/(p) — p| <$2^{1/2}$— 1 =ctn fw, the part of this line which lies in H_o is contained in s(p, l.fw). We show that h_Y approaches along this line. By substituting $(f(p)—p)y+p$ for x in the expression for A_n(,v, y), one obtains

A_n(x,y) = [(1 - ny) V 0]

(8) 17

^{r}n + 4 + 2(-

?n-

If P^s_n, then $f(p)$^l_n, and one can verify directly that (8) vanishes. Ifp>s_n, then $f(p)$ r_n, and again one can verify directly that (8) vanishes. Thus A_n(x, y) vanishes along that part of the line (7) which lies in H.

Solving (7) for *f(p),* we find that, along the given line,

/(p) = *(x-(l-y)p)ly,* and hence

p = */(/⊠(?)) = /*((x-(i *-y)p)ly)-* .

Therefore (if 0<y^|) *p=u(x, y).* Hence, if <x, y>satisfies (7), n^2, and l/(n+l) 1/n, then

h_0*(x, y) = (yn(n* +1) - n)g„(p) + ((»+!)-*yn(n* + l))g_{n+}j(p),

so that A_0(x, *y)* lies on the line segment joining $g_n(p)$ to $g_n+i(p)$- It follows that /z_0(x, y) approaches 0(p) as <x, y>approaches *p* along the line (7). Since each A_n vanishes on the part of this line lying in *H,* $h_r(x, y)$ also approaches ^(p) along this line.

Let s_m be any point of *E.* The definition of *f* shows that

|/(x)-x| 2 w(5)

seN

for all *x,* and from this it easily follows that

K-Jml 2 l/m«Jml = 2

sgN sgN

Hence

So the part of the line

(9)

that lies in H_o is contained in s(s_m, 1, fir). We show that approaches ^(s_m) as z -> s_m along this line. Substituting the value of *x* given by (9) into the expression for A_n, we obtain

A_n(x,y) = [(1-ny) V 0]

(10)

⊠n

If s_m<s_B, then /$_m$<r_m£/$_n$<r_n, and one can verify that (10) vanishes. If s_n<s_m, then l_n<r_n£l_m<r_m, and again one can verify that (10) vanishes. Thus, for n/m, A_n(x, y)=0 when <x, y>lies on the line (9) and in *H.*

If we take *n=m* in (10), we obtain

A_m(x,y) = [(1-wy) v 0]z_m.

Therefore A_m(x,y) approaches z_m=^(s_m)-^(s_m) along the given line.

Take any <x, y>e H_o satisfying (9), and take any *a* and *b* satisfying

(11) $a < s_m < b$.

Then *f(a)* ^l_m< |(r_m+/$_m$) < r_m £*f(b),* so that

(f(a)-s_m*)y+*s_m < *x* < *(f(b)—*s_m*)y+*s_m*;*

from which we deduce that

/(a) < (x-(1 *-y)*s_m*)ly* < *f(b).*

Since *f** is weakly increasing,

a = £*f*((x-(l-y)*$_{Sm}$*)ly)* £*f*(f(b»* = *b.*

Because *a* and *b* were taken to be any two numbers satisfying (11), we conclude that

$s_m = f*(.(x-(l-y)s_m)/y),$

whence it follows that $u(x,y)=s_m$. Thus

$h_0(x, y) = (yn(n+1) - ri)g_n(s_m)+((«+!) - yn(n + l))g_{,,+}i(s_m)$

when l/(n+l)^y^ 1/n. Consequently $h_0(x,y)$ approaches ^(s_m) along the line (9); so Ai(x, y) approaches along the given line.

We have shown that, for each *x e A,* there exists a line segment at *x,* lying in *s(x,* such that A_1(z)^-<^(x) as z->x along the line segment. We do not know that h_Y is bounded, but this is easily patched up. Choose a real number *B* such that, for all x e *A,*

—*B* < Re^(x) < *B,* —*B* < Im<£(x) < *B,* and set

h(z) = ([(ReA_1(z)) v (-B)] A B)+i([(Imft_1(z)) v (-B)] A B).

If we extend *h* to a bounded continuous function defined in *H,* and then restrict *h* to *Q,* we have the desired function.

(C) *Let d{t) be a weakly increasing, positive, real-valued function defined for* 0< *t* 1. *Then there exists a continuous, complex-valued function k defined in Q, with* |&(z)| $^\wedge 2^{1/2}$ *for all z e Q, such that for each a e* (0, 1] *and for each arc*

y £ { <x, y>: — 1 *x* 1, 0 < *y* g *a},*

{diameter y) 2: *d(a) implies {diameter k{y))* 2.

Proof. Let *p{x)=%d{t) dt* (0<x^l). Then *p* is positive, continuous, and strictly increasing, and *p{x)^\d{x).* Let ae(0, 1] be given. Since *p{xY^r* is uniformly continuous on each compact subset of (0,1], there exists ee(0,1] such that

(|a Xi 1 and |x_x-x_2| < e)

implies

IX*1)~^{1}~P(*2)~^{1}| 1-

Let *e{a)* be the supremum of all such *e.* Then *e{a)* is an increasing function of *a,* and

Qu x_x S 1 and |x_x-x_2| < e(a))

implies

Set ?(x)=fo *e(t) dt.* Then *q* is positive, continuous, and strictly increasing, and tf(x) = «(*)• Let *m{x)*=min *{p(x), q{x)}.* For <x, y>e *Q,* define

kfy) = sin *{2n/ym{y)), k_2{x, y)* = sin *{4nxlp{y)),*

k{x,y) = i(y) + i_2(x,y).

Now suppose that a e (0, 1] is given, and suppose that y£{ <x,y>: -l^x^l, 0<y^a} is an arc with (diameter *y)^d(a).* Choose z_1 = <x_1, y_x> and z_2=<x_2, y_2> in *y* so that lz_x—z_2| S *d{a).* Assume without loss of generality that y_2^yi. We can choose *a'* so that 0<|a'^y_1^a'^a. Since *m{a')^ d{a')^$ d{a),* and since *\ z_1—z_2\ ^d{a), we* must have either

(12) / *yi*-y_2| *m{a')*

or

(13) bi-Jal < *m{a')* and |x!-x_2| / *d{a').*

First assume that (12) holds. Here *m{y_2)^m{y_1)^.m{a'),* so

2?r/y_1m(y_1) *2jrly$_2$m{y_2),*

and we have *2tt 2tt 27r(j$_x$ffl(^i)-y$_2$m(y$_2$)) yzm(y$_2$) ypntyj* > 2n(y$_1$m(y$_2$)-y$_2$m(y$_2$)) = 2n(y$_1$-y$_2$) ~ *yiy2m(y$_1$)m(y$_2$)* yxy$_2$w(yx) > *27r^(fl/)* > 2tt >
= </sup> J1J2 <sup>

Thus, as <x, *y)* moves along y from <x$_x$, *y^* to <x$_2$> J2X we see that *2irlym(y)* varies over an interval of length at least *2tt,* and hence *k$_Y${y)* varies over the whole of the interval [—1,1]. Therefore (diameter *k(y))^2.*

Now assume that (13) holds. Then

4ttX! 4ttX$_2$ *p(yi)~ p(y 2)*

4t7

Xx *2 *p(yi) p(yi)*

x$_2$ x$_2$

P&2) p{yi)

r*ixi-x$_2$i*___________________________________ 1_____ *1* 1

L /»Oi) *p(yz) p(yi)* J

[i<)_ 11 1

Lx«) Xh) *p(yi)* J

S 477 11 -

1 1

p(y2) p(yi)

Now, *[y$_1$-y$_2$l<m(a')^q(a')^e(a'), so [p(y$_2$)$^{~1}$-p(yi)$^{~1}$[^j-* Therefore |47rx$_1$/p(y$_1$)-4irx$_2$/p(y$_2$)| 2tt, and we see that as <x, y>varies along y from <x$_x$, y$_x$> to <x$_{2>}$ y$_2$>, the quantity 4irx/*p(y)* varies over an interval of length at least *2tt,* so that *k$_2$(x, y)* takes on every value in the interval [—1, 1], Thus (diameter *k(y))^2.*

(D) *Let A ^X be a set of type F$_{a\{}$, and let</> be a bounded function of honorary Baire class 2(A, R^2). Then there exists a bounded continuous complex-valued function f defined in Q such that A is the set of curvilinear convergence of f and</> is a boundary function for f.*

Proof. Let *g* be the function of (A) and let *h* be the function of (B). For *t e* (0, 1], let

d$_t$(t) = sup {8 e (0,1] : *t,y$_2$Z t,* <x$_{15}$ y$_x$> e *Q,* <x$_2$, y$_2$> e *Q,* and
|<xi,yi>-<x$_2$,y$_2$>|< 8) implies |A(x$_1$,y$_1$)-/i(x$_2$,y$_2$)| *t},*

d$_2$(t) = sup {8 e (0,1] : (jx *t, y$_2$ t,* <Xx, jx>e *Q,* <x$_2$, y$_2$> e *Q,* and
|<Xx, yx>-<x$_2$, y$_2$>| < 8) implies |g(x$_{1;}$ *y$_1$)-g(x$_2$,* y$_2$)| S *t}, d(t)* = min <4(10,10-

Let *k* be the function of (C) for this *d(t),* and set *f(z)=h(z)+g(z)k(z) (z* e *Q).* We show that *f* is the desired function.

Suppose *x* e *A.* Then there exists an arc *y* at x, lying in *s(x,* 1, such that *h* approaches</> (x) along *y.* But *g(z)* approaches 0 through *s(x,* 1, $77) and *k* is bounded, so *g(z)k(z)* approaches 0 along *y.* Hence *f(z)* approaches</> (x) along y. Thus is a boundary function for /, and *A* is a subset of the set of curvilinear convergence of*f.* It only remains to show that if *x* is a point of the set of curvilinear convergence of/, then *x* e *A.* To show this, let *y* be an arc at *x* along which *f* approaches a limit. We may assume without loss of generality that y has an end point in { <x, 1 >: — 1 ^x^ 1}. By the properties of g, it will be enough to show that g approaches zero along *y.* Assume that g does not approach zero along *y.* Then there exists e e (0, 1] and there exists a sequence $\{z_n\}$ such that $z_n ey$ — {x}, z_n-+x as «->oo, and |g(z_n)| e for all n. Write $z_n = (x_n, y_n)$. Choose N so that n^N implies y_n<

For the time being, let n be a fixed integer greater than or equal to *N.* Set a = 4y_n/3. Let y be the component of y n Cl [S(J(tz), z_n)] that contains z_n. (Recall that *S(d(a),* z_n)={z : \ z-z_n\ <*d(a)*}.) Then

d(a) diameter / 2 *d(a),*

and, since *d(a) la,*

7 £ {<x, *y) : la* g *y £ a}.*

By the choice of *k,* there exist points p and q in y with | £(/»)—&(<7)| S2. We have *lp-q{* £2 *d(a)*< *dfila),* so, by the definition of J/t),

|A(p)-A(?)| *la* < |e.

Similarly,

\ *g(p)-g(Zn)*\ *la* < |e,

|g(?)-S(O| *la* <

Thus

I/(P)-/(9)| \ *g(p)k(p)˜g(.q)k(q)*\-\ *h(p)-h(q)*\

I *g(p)k(p)-g(z_n)k(p) +g(z_n)k(p) —g(z_n)k(q) +g(Zn)k(q)-g(q)k(q)*\ |g(z,,)|
k(p)-fc(?)|-k(/OI \ *g(P)-g(Zn)*\ -|£(?)l \ *g(<l)-g(Zn)*\-*le*

Z 2e-2^*ll2*^*le-2*^*ll2*^*le-l*$_e$ > *e.*

Note that \ *p*-z_n\ ^*d(a)*^*la=l*y_n, and similarly |^-z_n|^|y_n.

We have now shown that, for each n^*N,* there exist points p_n, $q_n ey$ with Ia>˜z,,| *il*y_n, kn-Znl *H*y_n, and |/(p_n)>«☒ But then p_n->x and q_n->x as n -> oo, so f does not approach a limit along *y.* This is a contradiction. We conclude that *g(z)* -> 0 along *y,* and hence that x e *A.*

(E)*Let A*^*C be a set of type* F_{ad}*, and let* <*f*>*be a bounded function of honorary Baire class* ^*2(A,* B^2*). Then there exists a bounded continuous complex-valued function f defined in D such that A is the set of curvilinear convergence of f and*</> *is a boundary function for f*

Proof. If A = 0, this is trivial. If A / 0, then we can assume, by making a suitable rotation of the disk, that <1, 0>g *A.* Let *G* = *D-S(&,* <|, 0» and let L = C — {<1,0>}. Because Q u X is homeomorphic with G u £, we see from (D) that there exists a bounded continuous complex-valued function f_Y defined in G such that

(i) A n L is the set of all points x e L such that/i approaches a limit along some arc at x, and

(ii) the restriction of</> to L is a boundary function for f_r.

Since G is closed relative to Z>, we can extend $f_\pm$ to a bounded continuous function f defined in D in such a way that /has <£«1, 0» as a radial limit at <1, 0>. This f will have all the desired properties.

(F) *Let* S^2 *denote the Riemann sphere, let A ^C be a set of type* F_{o6}, *and let*</> *be a function of honorary Baire class ^2(A,* S^2*). Then there exists a continuous function f: D-+* S^2 *such that A is the set of curvilinear convergence of f and*</> *is a boundary function for f*

(1)We suppose that

S'^{2} = {<x, *y, z)* g K^3 *:* x^2+y^2 + z^2 = 1}. We let

U = {<x, *y, z)e*S^2 *:* < $^{2\ 1}$

V = |<x, *y, z)* e S^2 : -1 *z* <

Zu = *y, z*> e s^2 *:* < *z £* 1|,

Z_v = | <x, *y, z*~>*e* S^2 *:* -1 z < -jkj"

We define mappings 0^: Z_v -> *U* and <J>$_y$: Z_v -> *V* by setting

<Mx, *y, z)* = <x(4z^2 -1), *y(4*z^2 - 1), z(4z^2 - 3)> «x, *y, z) g Zu)*

and

<D$_y$(x, *y,* z) = <x(4z^2 -1), *y(4*z^2 — 1), z(4z^2 - 3)> «x, *y, z)* g Z$_y$).

Then 0^ is a one-to-one continuous function from *Zu* onto *U.* Since Z$_a$ and *U* are each homeomorphic to the unit disk *D,* it follows from [7, Corollary 1, p. 122] that <>[/ is a homeomorphism of Z_v onto *U.* Similarly, O$_y$ is a homeomorphism of Z$_y$ onto *V,*

We define a continuous function O: S^2 -> S^2 by setting

O(x, *y, z)* = <D$_u$(x, *y, z),* < z 1,

0>(x, *y,* z) = <x, *y, -z},* z

<D(x, *y, z)* = <D$_v$(x, *y,* z), -1 z <

Notice that for each *p* e S^2, the inverse image set $^<$h$^{\sim1}$({p}) contains at most three points.

(II) Most of the results of Hausdorff[9] on real-valued Baire functions can easily be shown to hold also for functions taking values in A". We shall make free use of these results in this more general form.

(III)Now we proceed to the proof of (F). Let *N* be a countable subset of *A* such that the restriction of 0 to *A—N* is of Baire class *\|(A — N,* S^2*),* and let A=^ —*N.* It will be convenient to let F_a*(Ai)* denote the class of all subsets of A_t that are of type *F„* relative to A_{lt} and G_t{*A ^)* the class of all subsets of A_x that are of type G_6 relative to A. Since *U* and *V* are open subsets of S^2 and *U* u *V*=S^2, we see that $A_\pm$ n *j-^U)* e *F^A ^, A,-/-*^{1}*^) e G^AJ, and* A_1*-/*$^{\sim1}$*(F)^*A_1 n 0^{-1}(G). An elegant theorem of Sierpinski [8] now enables us to choose a set *X e F„(*A_1*)* n G$_d$(A) such that

A-0^{-1}(F) S *K £ Ai* n *^(U).*

Let *L* = *Ai-K.* Then *L e F^AJ n* G_f*(Ai).* Moreover, </>*(K)q U* and 0(L)£ *V.* Let/>! = <!, 0,0>, and define 0: *A* ->S^2 — (pj as follows. Set

^(x) = <D^(0(x)), ^(x) =

xeK, xeL.

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[9] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

If *xeN,* we let 0(x) be any element of $Z_V KJ Z_V$ for which d>(0(x))=0(x). This choice of 0(x) is always possible, because d>(Z_y u Z_y)2 *U'J V=S^2.* Let *</>o* be the restriction of 0 to *A ^KkjL.* I assert that 0_O is of Baire class

1(A» 5^2-{p_1}). Since S^2 - {pi} is homeomorphic to R^2, it will suffice to show that 0_O~ \G) e *F^Ai)* for every open set G£S'^{2}-{p_1}. But

0o \G) = A n *i/r\ G) = [Kn^G)]u[Ln ^>~\ G)]*

= *[Kn fWulZu* n G))] u [L n *<f>-\ ^Z_v c* G))] e *F^),*

so 0o is of Baire class 1(A, S^2—{pj). Now, $A_\pm$*A -N is of type G₆ relative to A, so (again using the fact that S²-{pj is homeomorphic to R²) we can extend 0o to a function 0_L of Baire class ^UA, S²-{pj). The existence of 0! shows that 0 is of honorary Baire class £2(A, S²-{pj). The range of 0 is contained in Z_v KJ Z_v, so that the values of 0 are bounded away from p_x. Thus, if we still think of S² - {pi} as corresponding to the plane R²,0 corresponds to a bounded function. By (E), there exists a continuous function f₁: D 5²-{pj} such that the values of are bounded away from p_n A is the set of curvilinear convergence of /₁₍ and 0 is a boundary function for f_r. Let f denote the composite function o Then f is continuous and <!>° 0=0 is a boundary function for f. It only remains to show that if x is a point of the set of curvilinear convergence of f, then x e A. Let y be an arc at x along which f approaches a limit, and let C(f_{lt} y) denote the cluster set of/i along y. Assume that x 0 A. Then does not approach a limit along y, so C(/i, y) contains infinitely many points. Now, 0 maps at most three points to any one given point, so 0(C(/_X, y)) contains infinitely many points. But 0(C(/i, y)) is the cluster set of/* *<I>° along y, and hence* f *does not approach a limit along y, contrary to our assumption. We conclude that* x e A *after all. This completes the proof of the theorem.*

The following questions remain open.

Problem 1. *If A is an arbitrary set of type F_{a6} in C, does there necessarily exist a continuous real-valued function in D having A as its set of curvilinear convergence!*

Problem 2. *If A ^C is a set of type F_{o6}, and if</> is a function of honorary Baire class ^2(A, R), does there necessarily exist a continuous real-valued function in D having A as its set of curvilinear convergence and</> as a boundary function ?*

Appendix. Some theorems concerning functions of Baire class 1 which take values on the Riemann sphere can be obtained by the technique used to prove (F). We use the notation set up in the proof of (F).

Theorem (a). *Let M be a metric space, and let</> : M -> S^2 be a function such that ^(G) is an F_a set for every open set G^S^2. Then (f> is of Baire class* l(Af, S^2).

Proof. Since *U* and *V* are open and *U u F*=S^2, it follows that the set

is F_a, the set *M*—is G_6, and By the theorem of

Sierpinski [8], there exists a set *K* that is simultaneously F_a and G_d such that

Let *L* = *M*—*K*. Then *L* is simultaneously F_a and G_6, and

W U, c: *V.*

Define 0: *M* S^2—{pj (where ^ = <1,0, 0» by setting

^(x) = 0 - i(^(x)), x g *K,* 0(x) = <D_y- ^(x)), x g *L.*

If *G* is an open subset of S^2 — {/h}, then

= *[KhWUZij* G))l u [£ n *n* G))],

so 0^{-1}(G) is an F_{ff} set. Since S^2—$\{p_1\}$ is homeomorphic to the plane, it follows that there exists a sequence $\{ip_n\}$ of continuous functions, each mapping *M* into S^2—$\{p_1\}$, such that pointwise on *M.* But then O(0_n(x))0(^(x))=^(x) for each fixed x g *M,* so</> is of Baire class 1(M, S^2).

A special case of Theorem (b) was proved (in effect) in [6, proof of Theorem 6] by means of a rather messy lemma (Lemma 3). Theorem (a) provides a proof that is both more general and more esthetically satisfactory.

Theorem (b). *Let M be a metric space, and let <£: M ->S^2 be a function. Then <f>is of honorary Baire class ^2(M, S^2) if, and only if, there exists a countable set N^M such that, for every closed set F^ S^2,</> $^{-X}$(F) — N is a G_6 set.*

Proof. The implication in one direction is trivial. Now assume that *N* is countable and that — *N is* a G_d set for every closed set *F*^S^2. Let</> $_0$ be the restriction of</> to *M*—*N.* Since S^2 is a subset of R^3, <$£_0$ is of Baire class ^1(M—*N*, R^3). Because *M*—*N* is a G_6 set,</> $_0$ can be extended to a function of Baire class

1(M, A^3). Now, <^(x) e S^2 except for only countably many x, so there exists some point *q* in the open ball enclosed by S^2 such that *q* is not in the range of fa. Define a mapping *P: R^3-{q}-+* S^2 as follows. If *a* g *R^3-{q},* let *L* be the ray with end point at *q* which passes through *a,* and let *P(a)* be the intersection point of *L* with S^2, Then *P* is continuous and *P(a)* = *a* for each *aeS*^{2}. Let ^Po<^₁. If G^S² is open, then 0⁻¹(G) = ^r¹(P⁻¹(G)), so that 0⁻¹((7) is an F_a set. Thus, by Theorem (a), i/j is of Baire class 1(M, S²). Moreover, if x $ N, then ^₁(x)^$_0$W = <£(x)g S^2, so that 0(x)P(^(x))^(x). Therefore</> is of honorary Baire class ^2(M, S^2).

An alternative proof of Theorem (b) could be given by combining Theorem (a) with the following result.

Theorem (c). *Let M be a metric space, E a G_6 set in M, <f>a function of Baire class* 1(£, *S^2). Then</> can be extended to a function of Baire class* l(Af, *S^2).*

To prove this, use the technique appearing in the proof of Theorem (a).

Finally, we note that a theorem proved by Bagemihl and McMillan for realvalued functions [1, Theorem 2] can be transferred to the Riemann sphere by means of our technique.

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9. July 1969 - Boundary functions and sets of curvilinear convergence for continuous functions

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Kaczynski, T.J. 1969. Boundary functions and sets of curvilinear convergence for continuous functions. *Trans. Am. Math. Soc.* 141:107-125.

MR0243078 Kaczynski, T. J. Boundary functions and sets of curvilinear convergence for continuous functions. Trans. Amer. Math. Soc. 141 1969 107.125. (Reviewer: J. E. McMillan) 30.62

Explanation by John D. Bullough

The author completes the investigation, initiated by Bagemihl and Piranian, of boundary functions of continuous complex-valued functions defined in the open unit disk D. the set of curvilinear convergence A of such a function f is defined to be the set of those e^{iT} at which f has a finite or infinite limit along some open Jordan arc lying in the disk and having one endpoint at e^{iT}. A boundary function of f is a function t defined on A such that each $t(e^{iT})$ is one of these limit values. The author proved that t differs from some function of the first Baire class at at most countably many points, and McMillan proved that A is of type F(sd). By means of an intricate construction, the author proves that for any set A on the unit circle of type F(sd), and for any function t defined on A that differs from some function of the first Baire class at at most countably many points, there exists a continuous complex-valued function f defined in D having A as its set of curvilinear convergence and having t as its boundary function. The author points out the the problem remains open for real-valued functions.

Article by Ted

10. Nov 1969 - The Set of Curvilinear Convergence...

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Kaczynski, T.J. 1969. The set of curvilinear convergence of a continuous function defined in the interior of a cube. *Proc. Am. Math. Soc.* 23:323-327.

MR0248339 Kaczynski, T. J. The set of curvilinear convergence of a continuous function defined in the interior of a cube. Proc. Amer. Math. Soc. 23 1969 323.327. (Reviewer: J. E. McMillan) 30.62

Explanation by John D. Bullough

The set of points of the unit circle at which a continuous complex-valued function in the open unit disk has limits along curves (asymptotic values) is of type F(sd) and, in general, has no other properties. The author shows that for continuous complex-valued functions defined in a cube, this set of "curvilinear convergence" does not even need to be a Borel set. He asks whether such an example can be given for real-valued functions.

Article by Ted

THE SET OF CURVILINEAR CONVERGENCE OF A CONTINUOUS FUNCTION DEFINED IN THE INTERIOR OF A CUBE

T. J. KACZYNSKI

Let Q be an open connected set in a finite-dimensional Euclidean space, and let f be a function mapping Q into another finite-dimensional Euclidean space. We define the *set of curvilinear convergence of* f to be

*{pE:*boundary of Q: there exists a simple arc 7 with one endpoint at p such that 7 — $\{p\}$ CQ and/(fl) converges to a finite limit as v—$+p$ along 7}.

J. E. McMillan[1] has shown that if Q is an open disk in the plane and if f is continuous in Q, then the set of curvilinear convergence of f is of type F_o*$.* In this paper we prove that there exists a bounded continuous complex-valued function /, defined in

[1] M. H. A. Newman, *Elements of the topology of plane sets of points,* Cambridge University Press, 1961.

the interior of a three-dimensional cube, such that the set of curvilinear convergence of f is not a Borel set. Thus McMillan's theorem does not generalize to three dimensions. However, the following question remains open.

Problem. *Does there exist a continuous real-valued function f, defined in the interior of a three-dimensional cube, such that the set of curvilinear convergence off is not a Borel set?*

Let

R be the set of real numbers

R^n — w-dimensional Euclidean space

Q= l(x,y)Gi?^{2}:0<ySland

☒K={(x,y, z)Gi?^{3}:0<y^l, -l^x^l.and -l^z^l}

<2° = interior of Q

K° = interior of K.

Let Q again represent an open connected subset of R^n. If/: Q—>R^m is a function, we shall say that *a£*R^m is an *asymptotic value* of / iff there exists a continuous function *v:* [0,1)—such that dist(fl(/), R^n — Q)—>0 and/(fl(£))—>& as *t*—>1. (Note that a limit approached by / along a path which tends to 00 may or may not be an asymptotic value by our definition.) We say that a is a *point asymptotic value* of / (at *p)* iff *v* can be chosen so that, as *t*—*1, *v(t)* approaches

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a point —Q. Because of the result of [8], the set of curvilinear convergence of f is *{pE:*R^n — has a point asymptotic value at *p}.*

Lemma. *There exists a continuous complex-valued function s defined in*

{(x,y) G R^2*: y*> 0},

with | six, y) | 1 *for all x and y, such that s has the following property.*

Let E be the set of all asymptotic values of s that are real and lie in the interval (— 1, 1). *Then E is equal to the set of all point asymptotic values of s that are real and lie in* (— 1, 1), *and E is not a Borel set.*

Proof. Let A be an analytic subset of R that is not a Borel set. (This exists [7, p. 254].) We see from the paper of Kierst[2] that there exists a holomorphic function h defined in *{z: z* is a complex number and | *z*| < 11 such that h omits the three values *—i, i,* oo and *A*|*J {* — *i, i*] is the set of all (finite) asymptotic values of *h.* The function h is then normal [5, p. 53], so, as pointed out by McMillan [6, p. 311], it follows from Theorem 1 of[3] that *A*|*J {—i, i*| is just the set of all (finite) point asymptotic values of *h.* We now obtain the desired function by setting

s(x, *y)*

Z?((l — *y)*e^{ix}*)*

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[2] P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.,* 4 (1957) 155-156.

[3] F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math, J.,* 8 (1961) 201-207.

(0<y^ 1),
1 + | *h(fl - y)e^{ix}) /*
/z(0)
(y 1).
$(x, y) = ——————————- i——- r
1+| *h(O) /*

Remark. Since the theorem we want to prove has nothing to do with meromorphic functions, it is unfortunate that the proof of the lemma depends on the theory of meromorphic functions. This can be avoided. The lemma can be proved by using [7, Theorem 113, p. 216], [1, Theorem 2, p. 179], and the methods of[4], but this involves a messy construction, so we omit the details.

Theorem. *There exists a bounded continuous complex-valued function f defined in K° such that the set of curvilinear convergence off is not a Borel set.*

Proof. Let 5 and *E* be as described in the lemma, and set g(x, y) = s(x/y, y) for (x, y)GQ- The reader can verify that *E* equals the set of all real point asymptotic values of *g* at the point (0, 0) which lie in the interval (— 1, 1). For each /G(0, 1], define

tZo(O = sup{a G (0, 1]: *((x, y)* G *Q, (%', y^f) G Q, y t, y' t,* and | *(x, y)* — (x', *y')* | < 5) implies | *g(x, y) — g(x', y') / t}, d(t)* = min{|d_o(JO, 1^-

By statement (C) of[5], there exists a continuous complex-valued function *k* defined in Q, with |fe(x, y) | ^2^{1/2} for all (x, *y)GQ,* such that for each oG(0, 1] and for each arc

7 C {(x, y): — 1 x 1 and 0 < y g *a},* (diameter 7) ^d(a) implies (diameter £(7)) ^2.

Let *f* be the function with domain *K°* defined by /(x, y, 2) = (g(x, y) —2)&(x, y). We note that the following inequality holds for any three points (x, y, 2), (x', y', 2'), (x", y", 2") in *K°:*

I/O', *y', z') -f(x",y",z")* |

= | (g« *y') - z')k(x', v') - (g(x, y) - z)k(x', y') + (g(x, y) ~ ^k(x', y') - (g(x, y) - z)k(x", y") + (g(x,y) - z)k(x",y") - (,g(x",y") - z")k(x",y")* / I *g(x, y)* - z I I *k(x', y') - k(x", y")* I

(1) - / *k(x',* y) | |g« *y') - z' - g(x, y)* + z I

- / *k(x", y")* 11 *g(x, y) — z — g(x", y") + z" /*

I *g(x, y)* - z 11 *k(*', y'} - k(x", y")* I

- 2 I *g(x',y ') - g(x, y)* I - 2 I g(x, *y) - g(x", y") /*

- 2 | *z* — z' | — 2 | z" — z | .

LetZ= {(0, 0, 2): — 1 <2 < 1}, and let T be the set of curvilinear convergence of *f.* We wish to show that rP\L= {(0, 0, 2):2G^}.

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[4] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17 (1931) 283-295.

[5] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17 (1931) 283-295.

Suppose &G£- Then there is an arc 7 with one endpoint at (0, 0) such that 7— {(0, 0)} CQ° and *g* approaches *b* along 7. Let

7' = {(*, y, *b): (x,* y) G 7}.

Then g(x, y)—2—>0 as (x, y, 2)—*(0, 0, *b)* along 7'. Thus, since *k* is bounded,/(x, y, 2)—>0 along 7', so (0, 0, &)GLHL.

Now let us assume, conversely, that (0, 0, &)GrO£ and deduce that &G£- Let 7' be an arc with one endpoint at (0, 0, *b)* such that 7'— {(0, 0, *b)} QK°* and / approaches a limit along 7'. Let

7 = {(x, y) G R^{2}' (x, y, 2) G 7' for some 2}.

Then 7 is a (not necessarily simple) arc with one endpoint at (0, 0) and 7— {(0, 0)} CQ°. I assert that g(x, y) —2 approaches 0 along 7'.

Assume this is false. Then there exists $c > 0$ and there exists a sequence of points {(x_n, y_n, z_n)} $*_{=1}$ in 7' — {(0, 0, *b)*} such that

(x_n, y_n, Z_n) (0, 0, *b)* as n 00

and | g(x_n, y_n) —z_n\| for all *n*. Let 8>0 be chosen so that whenever (w, *v, w)£y'*, (x, *y, z)Ey',* and *v, y^b,* then |w—*z*\ <|e. Let *N* be chosen so that *n N* implies y_n < min {3e/32, 33/4, 3/4}.

For the present, let *n* be a fixed integer greater than *N.* Set *a* = ^y_n/3*.* There exists an arc 7* contained in

7 *C*\ {(x, *y) G* R^{2}*. | (x, *y)* — (x,,, y_n) / *d(a)*\

joining (x_n, y_n) to a point on the circle of radius *d(a)* about (x_n, y_n)*.* Clearly (diameter 7*) (a), so (diameter *k(y*))^2.* Choose points

(x» , yn), (xX, y") in y* with *k(xj, yn)-k(x'n', y")*| ^2. Choose *Zn , z'n* so that (x_n', y_n', *Zn)* and (x,,, y,,', *z,,)* are in 7'. It is easy to check that Ja^y/ <8 and |a^y"<8, so

(2) / z_n — *Zn* | < je and | z_n" — z_n | < |e.

Moreover, since | (x_n', y_n') — (x_n, y_n)| ^*d(a)* g *%d^a),* we have

g(Xn, yn) - g(%n , yn)	 <	e;

and similarly

g(.r,,", *yn') - g(x_n, y_n)* I <	e.

Combining these inequalities with (1) and (2), we get

/ *f* (x_n *J yn y* Z_n *) f* (x_n , y_n , z_n *)* / >

/ *g(%n, yn)* Z_n | | ^(x_n , y_n) *k*(x_n , y_n) / £ = 2fi € — €.

But y_n', *yn* ^y_n/3*,* so (x_n', y_n', *z,,)*—»(0, 0, *b)* and (x", y_n", 2/') —>(0, 0, *b)* as n—»oo ; hence *f* cannot approach a limit along 7', which is a contradiction. We conclude that g(x, *y)— z*—>0 as (x, y, *z)* -^(0, 0, *b)* along 7'.

It follows immediately that g(x, *y)-^b* along 7, so *b^E.* We have now shown that rnz = $\{(o,0,z):zE \pounds\}$.

Thus $TC\backslash L$ is not a Borel set. Hence T is not a Borel set; for if it were, then TPiL would also be a Borel set.

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Ted's Work from his Parents Home in Illinois

11. Problem 786

January, 1971
https://doi.org/10.2307%2F2688865
By T. J. Kaczynski, Lombard, Illinois.

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into $n2$ little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

12. A Match Stick Problem

November–December 1971
https://doi.org/10.2307%2F2688646

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Problem 786. [January, 1971] Proposed by T. J. Kaczynski, Lombard, Illinois.

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into $n2$ little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

I. Solution by Richard A. Gibbs, Hiram Scott College, Nebraska.

A necessary and sufficient condition that a solution exist is that n be even.

Sufficiency is easy. If $n = 2k,$ consider the cardboard as consisting of $k2$ 2X2 squares. Simply place a match on each of the four segments adjacent to the center point of each 2X2 square.

For necessity, assume a solution exists for an nXn sheet of cardboard. To each unit square correspond the point at its center. Connect two points if their corresponding squares share a match. By the hypotheses, every point will be joined to exactly two others. Therefore, according to a basic result of Graph Theory, the resulting graph will be a collection of disjoint cycles. Each cycle will enclose a polygonal region whose sides are either horizontal or vertical line segments. Consequently, since the length of each segment is an integer, the area of each polygonal region will be an integer. By Pick's theorem (a beautiful result familiar to anyone who has played with a geo-board) the area of the 2th polygonal region is

$A = \%Pi + li - 1$

where there are Pi points on the perimeter and li points in the interior of the 2th polygonal region. Since each area is an integer, each Pi is even. As each point is on exactly one perimeter, the sum of the Pi is the total number of points, $n2$. Hence n is even.

II. Solution by Richard L. Breisch, Pennsylvania State University.

A generalization of the stated problem will be demonstrated. Let the cardboard be an m X n rectangle. The problem of covering the cardboard in the stated manner has a solution if and only if m and 72^2, and m and n are not both odd.

An alternative representation of the problem will be used to demonstrate this. Consider the mXn array of the center points of the little squares. If two edge-adjacent squares have a match on their mutual edge, connect the centers of these squares with a line segment. Since each little square has exactly two of its sides covered by matches, in the alternative representation, there are exactly two line segments from each point in the array. Hence each connected set of line segments forms a polygon, and the mXn array is covered by a collection of polygons. Each polygon must have an even number of horizontal segments and an even number of vertical segments. Since there are m -n segments, m and n cannot both be odd integers.

Suppose m is even. Then the mXn array can be covered with $m/2$ rectangular polygons each of which has dimensions 1 segment by n segments. The arrangement of matches in the original representation is easily derived from this representation.

Also solved by Dan Bean, Dave Harris and E. F. Schmeichel (Jointly), College of Wooster, Ohio; Thomas A. Brown, Santa Monica, California; Melvin H. Davis, New York University; Roger Engle and Necdet Ucoluk (jointly), Clarion State College Pennsylvania; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Heiko Harborth, Braunschweig, Germany; Herbert R. Leifer, Pittsburgh, Pennsylvania; Joseph V. Michalowicz, Catholic University of America; George A. Novacky, Jr., University of Pittsburgh; J. W. Pfaendtner, University of Michigan; Sally Ringland, Shippenville, Pennsylvania; Rina Rubenfeld, New York City Community College; E. P. Starke, Plainfield, New Jersey; and the proposer.

Ted's Work as a Montana Hermit

Never published new ground?

Ted went back to playing around with pure math equations briefly in his cabin in Montana. He even wrote the kind of math paper you would submit to a journal, but never sent it anywhere.

Here's how Ted explained the paper in relation to his other work:[2]

> (Ca) FL #80, letter from me to my parents, Spring, 1964, p. 1: "It's a good thing I didn't follow Piranian's suggestions about how to attack the problem, or I never would have solved it!"
>
> Piranian urged me to prove (a) that every continuous function in the disk admits a family of disjoint arcs, and to deduce from this (b) that every boundary function for a continuous function can be made into a function of the first Baire class by changing its values on at most a countable set. (The terminology is explained in F. Bagemihl and G. Piranian, "Boundary Functions for Functions Defined in a Disk," *Michigan Mathematical Journal*, 8 (1961), pp. 201–207.)
>
> I maintained that it would be much easier to prove (b) by examining inverse–image sets, and I even suggested that (b) might then be used to prove (a). And that's how it turned out. I did prove (b) within three months or so by using inverse–image sets. The proof of (a) was vastly more difficult. I didn't succeed in proving (a) until two decades later, and I had to use (b) in order to do it. The proof of (a) has not been published.

And here's a glimpse into Ted's headspace when writing it, from a journal entry at the time:[3]

> Ever since seeing how the Trout Creek area has been ruined I feel so much grief whenever I am sitting quietly, or when I am walking slowly through the woods just looking and listening, that I have to keep occupied almost all the time in order to escape this grief. That was my favorite spot. Whoever has read my notes knows very well what the other causes have been. Where can I go not to enjoy in peace nature and the wilderness life? — which are the best things I have ever known. Even in the officially designated

[2] Truth versus Lies (Original Draft)
[3] Journal #1 of 4 from Series VII (1984-1986)

“wilderness” there must be the continued noise of airplanes, especially the jets, since I know that planes are permitted to fly over the Bob Marshal and Scapegoat wildernesses. Are there fewer planes there than here. Maybe, maybe. Perhaps one of these days I’ll go and find out. But so many times I’ve gone looking for a place where I can escape completely from industrial society, and always . . . [three dots in the original] well, I’m very discouraged. So, I’ve been playing around with mathematics a good deal lately. It’s a rather contemptible game, but while I’m involved in it, it enables me to escape from my grief.

13. Four-Digit Numbers that Reverse Their Digits When Multiplied

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FOUR-DIGIT NUMBERS THAT REVERSE THEIR DIGITS WHEN MULTIPLIED

T. J. KACZYNSKI

If n <i-2 is an integer and ag, . .. * a^ are integers satisfying 0 n for i « 0,1, • • • ,h ,

then we let ... , ap ag)$_n$ denote the number

^{a}j^{aJ} • Whenever we write a symbol of the form (a$_h$, ... , a., ag)$_n$ * it is to be understood that

0 a_£ n for i — Oy1f.««fh so that ^{a}h® • • • > ^{a}l ag are the digits of the number (a$_h$, ... , *p ag)$_n$ ^{in} base n notation.

If k is an integer and 1 k < n , we say that (a^, ... , ap ^{a}o)$_n$ i® *reversible for* n, k if and only if a^ / 0 and kCa^, ... , ap a$)$_n$ a (ag, a^$_t$ ... , ^{a}h)$_n$ • Reversible numbers have been studied in[1], [2j,[2]. The purpose of thia paper is to construct a rather involved family of 4-digit reversible numbers that illustrates the complexity of the reversible number problem. We use the abbreviation RN for "reversible number".

Sutcliffe [jj showed that there exists a 4-digit RN for any base n >3 • Let d be any divisor of n

(possibly n itself) with d "3 , and set t = n/d and k = d-1 . Then

k(t, t-1, n-t-1, n-t)$_n$ = (n-t, n-t-1, t-1, t)$_n$.

(This family of $^{in\ Let\ us}$

refer to a RN of this type as a *Sutcliffe* RN. Note that the Sutcliffe reversible number (t, t-1, n-t-1, n-t)$_n$ is equal to (n+1)(t-1, n-1, n~t;$_n$.

At least two other types of 4-digit RNs may exist for certain values of n. ,

If (a,b,c)$_Q$ is a J-digit RN for n, k , and if a+b £ n-1 and b+c < n-1 , then (n*1)(a,b,c)$_n$ is a 4-digit RN for n, k • (For instance, 4X (2,5,9) =

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[1] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat, Acad, Sci. U. S. A.>* 4 (1955) 379-382.

[2] S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund, Math.,* 17

$(9,5»2)_{17}$; multiplying by 18 yields $4X(2,7,14,9)_{17}$ ª (9,14,7,2)•)

If $(a,b,c)_n$ is any solution of the system of conditions

$k(a,b,c)_n = (c\text{-}1, b+1, a)_Q$,

(1)

a+b £ n-2 , b+c >☒ n , a 0 ,

then $(n+1)(a,b,c)_Q$ is a 4-digit RN for n, k , as can

be verified by computation. We note th*at* 4^/7 Jw /•

from a solution of (1) can never be a Sutcjlffe RN for n, k , because if t = n/(k+1) then $(t\text{-}1, n\text{-}1, n\text{-}t)_Q$ cannot satisfy (1).

One family of solutions of (1) can be obtained by taking any integers u - 1 and k 3 and setting $n = u(k^2\text{-}1)+k$, a = (k-1)u , b a (u(k+1) + 1)(k-2) , c = (uk+1)(k-1) . Observe that the corresponding 4-digit RN is $(n+1)(a,b,c)_n = (k\text{-}1, k\text{-}3, k\text{-}1)_n(u,uk+1)_n$, and that $(u,uk+1)_n$ is a 2-df g-'RN for n, k .

Sutcliffe [J] showed that there exists a 2-digit RN in base n notation if and only if n+1 is not prime. It was shown in [1] that there exists a 3-digit RN for n if and only if n+1 is not prime. This directs our attention to 4-digit RNs in the case where n+1 is prime.

Does (1) ever have a solution when n+1 is prime? The answer is yes. With n+1 = 59 we have 19X(2,41,52)53 = (51,42,2)^ , which yields 19 *(2,44,35,52)^ = (52,35,44,2)53 .

Do there exist infinitely many such examples? The answer is again yes.- Let s be any nonnegative integer, take k « 19 , n = 5&+360s , a = 2+17a , b = 41+260s , c = 52+323s , and we have a solution of (1). By Dirichlet's Theorem, there are infinitely many positive integers s for which n+1 « 59*360s is prime.

However, all these solutions are in a sense

isomorphic; w© do not regard them as essentially different. What we really want to show is this:

There exist infinitely many positive integers k *having the property that there exist integers* n, a, b, c *for which* n+1 *is prime and the system of conditions* (1) is *satisfied.*

This is our main result. To prove it, set

$f(x) = 41067x^2$ - UOhx ☒ 9 $g(x) = 10179x^2 - 222x$ ☒ 1 .

The discriminant of g(x) is 8%8 » 2^-1071 , not a square, so g(x) has no linear facWr with rational coefficients. Therefore f(x) and g(x) have no nonconstant common factor with rational coefficients. Consequently there exist polynomials p(x) and q(x) , with rational coefficients, such that p(x)f(x) •> q(x)g(x) * 1 . Let d>0 be the product of the denominators of all the fractions that appear as coefficients of p(x) and q(x) , and let pfx) » dp(x) and Q(x) = dq(x) • Then P(x) and Q(x) have integer coefficients and P(x)f(x) ☒ Q(x)g(x) = d .

Let k be any number of the form k « 1l7yd-2 , where y is a positive integer. Let D = yd and let v be the greatest common divisor of f(D) and g(D) . Then v divides D a

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(1931) 283-295.

yP(D)f(D) ⊠ yQ(D)g(D) . Since v divides g(D) it follows that v divides 1 . Thus f(D) and g(D) are relatively prime.

By Dirichlet*s Theorem, we can choose a positive integer t for which f(D)t * g(D) is prime. Set

n = f(D)t ⊠ g(D) - 1 = (4iO67D^2-UO4D*9)t*1O179D^2-222D ,

u = 1JD , r = 2(u-1) , m = 117Dt-t*29D = (9u-l)t+29D ,

U = 3u-1 , R = 3r+1 a 6u-5 = 78D-5 , M = 9m*1 ,

w = 9rm*Jm*r .

We compute

k = 117D-2 a 9u-2 = 3U+1 , n a MU*1 , MR = 3w*1 .

Modulo 9u-1 we have the following congruences:

nR+w a (MU*1)(6u-5)*9rm*3m+r

« (27mu+3u-9m)(6u-5)⊠18mu-15m*2u-2

= (3»*3u-9m)(6u-5)*2ia-15m>2u-2

a 18u^2-13u-36mu+17m-2 = -2u*13m-3

= -26l>377D-3 = 351D-3 « 3(117D-1) = 3(9u-1)

= 0 (mod 9u-1) .

Thus nR+w is divisible by 9u-1 • Choose an integer c so that (k+1)c = (9u-1)c *a* nR+w . Set S = knR-(k^2-1)c-1 . Because (n+1)R = (MU+2)R - 1 (raod 3) , we see that k-1 = 3U divides MIl[(n+1)R-l]. Thus

Sn-R+1 = (kn^2-1)R-(k^2-1)nc-(n-1)

E(n^2-1)R-(n-1) = MU[(n+1)R-1 j 5 0 (mod k-1) .

Choose an integer b so that (k-1)b = Sn-R+1 . Sot a - kc-Rn . We then have

(2) kc = Rn+a

(3) kb+R = Sn+b+1

(4) ka+S = c-1

We must show that certain inequalities are satisfied. Clearly 2 k <i n , c >2 , 2 R k-1 . Thus (k^2-1)c = 3U(k+1)c = 3U(nR+w) < 3UnR+UMR < 3UnR+nR $_3$ knR < kn(k-l) < (k^2-1)n .So 2 < c < n .

Observe that R-1+U < 3U 2(R-1)*U . Adding 3U(k*1)c = 3V(nR+w) to thia inequality gives

< *J* 3U(nR+w)4-R-1+U < 3U(k+1)c^^^3U(nR+w)+2(R-1)+U , (k-1)nR*R-1+MRU <(k^2-1)c+k-1 -4 (k-1)nR*2R-2+MRU , (k-1)nR>nR-1 < (k^2-1)c*k-1 (k-1)nR+nR+R-2 ,

1 <(k^2-1) c-knR+k+1 <R ,

k-R<.S < k-1

Thus 2 < S < k-1 (from which we see that b >0) and

(5) S+R > k+1 .

Also, (k-1)b = Sn-R+1 Sn (k-2)n^ ﬀe^so that b<T < ⊠^'n = n-1 , and b+1 < n .

p

Note that (k+1) C n , so that (k+1)c = nR+w > 2 (k+1) and c-1 > k >S • Thus ka = c-1-S > 0 , so that a 0 .

From (J) and (4), we find k(a+b)+R+S = Sn+b+c <^ (S+2)nt~kn • Therefore a+b < n . Suppose a+b = n-1 • Then from (4) and the definition of b we have (k-1)(n-1) = (k-1)(a+b) = S(n-1)+c-a-R . Consequently n-1 divides c-a-R . But c >ka by (4), so n-1 >c-a-R >(k-1)a-R>0 . This contradiction shows that a+b n-2 .

From (3) and (5) we see that (k-1)(b+c) s Sn-R+1+(k+1)c-2c = (S+R)n+w+1-R-2c (k+1)n+w+1-R-2c >

(k-1)n+w+1-R . But 3R MR = 3w+1, so that R < w+1 . Therefore b+c >n •

Equations (2), (3)» (4), together with the inequalities we have just proved, show that $(a,b,c)_n$ satisfies (1). ©

In the foregoing argument there is no need to restrict ourselves to the case where n+1 is prime, so the construction also yields many 4-digit RNs for composite values of n+1 .

We hooe to publish at a later date a more general treatment of reversible numbers, in which we shall prove (among other tilings) that if n*1 is prime, then every 4-digit RN for n is either a Sutcliffe RN, or of the form (n+DfajbjC) , where $(a,b,c)_n$ is a solution of (1).

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14. Handwritten Math equations and procedures

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