

The Mathematical Work of Ted Kaczynski

Ted Kaczynski

Mar 16, 2020

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Introduction by Jørgen Veisdal

The Mathematics of Ted Kaczynski

Disclaimer: As should be fairly evident, this essay is in no way meant to glorify Ted Kaczynski. Rather, it was written with two goals in mind: 1. To orient to reality some of the myths of Kaczynski’s ”genius” and 2. To illustrate yet another example of a mathematician whose abstract endeavours ultimately defeated him.

Before terrorist Theodore John Kaczynski (1942-) began sending mail-bombs to faculty members at various American universities, he had a promising career in mathematics. In particular, between 1964–69, he published a total of six single-authored research papers in renowned mathematical journals, including *The American Mathematical Monthly* and *Proceedings of the American Mathematical Society*.

The young Kaczynski did work in analysis, specifically geometric function theory in the narrow subfield of boundary values of continuous functions. The purpose of this article is to give an introduction to this work.

Education (1958–67)

Kaczynski grew up in Illinois, where he attended Sherman Elementary School and Evergreen Park Central Junior High school. At the age of 10 years old, his IQ was evaluated to be 167, and so he skipped the sixth grade (Chicago Tribune, 2017), an event later described as pivotal to his development (Chase, 2004):

“Previously he had socialized with his peers and was even a leader, but after skipping ahead he felt he did not fit in with the older children and was bullied.”

Harvard University (1958–62)

Kaczynski entered Harvard University in 1958 at the age of 16 years old. A mathematical prodigy since he was a child, he was described by other undergraduates as “*shy*”, “*quiet*” and “*a loner*” who “*never talked to anyone*” (Song, 2012):

“He would just rush through the suite, go into his room, and slam the door [...] When we would go into his room there would be piles of books and uneaten sandwiches that would make the place smell”

His personality notwithstanding, Kaczynski's talent was however still recognized among his Harvard peers, one of which in 2012 stated:

“It's just an opinion — but Ted was brilliant [...]. He could have become one of the greatest mathematicians in the country”

Kaczynski graduated Harvard with a B.A. in mathematics in 1962. When he graduated, his GPA was 3.12, scoring B's in the History of Science, Humanities and Math, C in History and A's in Anthropology and Scandinavian (Stampfl, 2006).

University of Michigan (1962–67)

With an IQ of 167, Kaczynski had been expected to perform better at Harvard. After graduating, he applied to the University of California at Berkeley, The University of Chicago and the University of Michigan. Although accepted at all three, he ended up choosing Michigan because the university offered him an annual grant of \$2,310 and a teaching post. The “darling of the math department”, he would graduate from the University of Michigan in 1964 with a M.Sc. in mathematics and markedly improved grades — 12 A's and five B's, which he himself later attributed to the standing of the university:

“[My] memories of the University of Michigan are **not** pleasant [...] The fact that I not only passed my courses (except one physics course) but got quite a few A's shows how wretchedly low the standards were at Michigan”

Nonetheless, as the story goes, while there once a professor named George Piranian told his students — including Kaczynski — about an unsolved problem in boundary functions. Weeks later, Kaczynski came to his office with a 100-page correct, handwritten proof. Kaczynski graduated with a Ph.D. in mathematics in 1967. His dissertation, entitled simply “*Boundary Functions*” regarded the same topic as his proof of Piranian's problem. His doctoral committee consisted of professors Allen L. Shields, Peter L. Duren, Donald J. Livingstone, Maxwell O. Reade, Chia-Shun Yin. Every professor approved it. His supervisor Shields later called his dissertation

“The best I have ever directed”

An additional testament to its quality was it being awarded the Sumner Myers Prize for the best mathematics thesis of the university, accompanying a prize of \$100 and a plaque in the East Quad Residence Hall entrance listing his accomplishment. Of the complexity (or perhaps narrow implications) of his dissertation, one of the members of his dissertation committee, Maxwell Reade, said

“I would guess that maybe 10 or 12 men in the country understood or appreciated it”

Another, Peter Duren, stated

“He was really an unusual student”

Kaczynski at UCB in 1967 (Photo: Wikimedia Commons)

University of California, Berkeley (1967–69)

In late 1967, at 25 years old Kaczynski was hired as the youngest-ever assistant professor of mathematics at the University of California at Berkeley. There, he taught undergraduate courses in geometry and calculus, although with mediocre success. His student evaluations suggest that he was not particularly well-liked because he taught “*straight from the textbook and refused to answer questions*”.

He resigned on June 30th, 1969 without explanation.

Work (1964–69)

Wedderburn’s Theorem

Kaczynski’s only published paper relating to topics other than boundary functions was his first journal paper, written before he started his Ph.D. It is entitled:

- **Kaczynski, T.J. (1964).** “Another proof of Wedderburn’s theorem”. *The American Mathematical Monthly* 71(6), pp. 652–653.

The paper concerned a 1905 result of Joseph H. M. Wedderburn that every finite skew field is commutative. His paper provided a group-theoretic proof of the theorem, which had previously been proved at least seven times.

Boundary Functions

Kaczynski’s Ph.D. dissertation concerned boundary values of continuous functions and was entitled, simply

- **Kaczynski, T.J. (1967).** *Boundary Functions*. Ann Arbor: University of Michigan.

Let H denote the set of all points in the Euclidean plane having positive y -coordinate, and let X denote the x -axis. If p is a point of X , then by an arc at p we mean a simple arc γ , having one endpoint at p , such that $\gamma = \{p\} \cup H$. Let f be a function mapping H into the Riemann sphere.

Boundary Functions By a boundary function for f we mean a function φ defined on a set $E \subset X$ such that for each $p \in E$ there exists an arc γ at p for which $\lim_{z \rightarrow p, z \in \gamma} f(z) = \varphi(p)$

Kaczynski's dissertation begins by re-proving a theorem of J. E. McMillan which states that if $f(H)$ is a continuous function mapping H into the Riemann sphere, the set of curvilinear convergence of F (the largest set on which a boundary function for f can be defined) is of a certain type. This proof also shows that if A is a set of the same type in X , then there exists a bounded continuous complex-valued function in H having A as its set of curvilinear convergence. The dissertation contains two additional new proofs related to boundary functions, and a list of problems for future research. Of the results, Professor Donald Rung later stated:

What Kaczynski did, greatly simplified, was determine the general rules for the properties of sets of points of curvilinear convergence. Some of those rules were not the sort of thing even a mathematician would expect.

Kaczynski would publish five journal papers related to the work from his dissertation between 1965–69:

- **Kaczynski, T.J. (1965).** “Boundary functions for functions defined in a disk”. *Journal of Mathematics and Mechanics*. 14(4), pp. 589–612.
- **Kaczynski, T.J. (1966).** “On a boundary property of continuous functions”. *Michigan Math. J.* 13, pp. 313–320.
- **Kaczynski, T.J. (1969).** “The set of curvilinear convergence of a continuous function defined in the interior of a cube”. *Proceedings of the American Mathematical Society* 23(2), pp. 323–327.
- **Kaczynski, T.J. (1969).** “Boundary functions and sets of curvilinear convergence for continuous functions”. *Transactions of the American Mathematical Society*. 141, pp. 107–125.
- **Kaczynski, T.J. (1969).** “Boundary functions for bounded harmonic functions”. *Transactions of the American Mathematical Society*. 137, pp. 203–209.

The Distributivity Problem

The only other trace of Kaczynski in a mathematical journal is two notes in the *American Monthly* in 1964 and 65:

- **Kaczynski, T.J. (1964).** “Distributivity and $(-1)x = -x$ (Advanced Problem 5210)”. *The American Mathematical Monthly*. 71(6), pp. 689.
- **Kaczynski, T.J. (1965).** “Distributivity and $(-1)x = -x$ (Advanced Problem 5210, with Solution by Bilyeu, R.G.)”. *The American Mathematical Monthly* 72(6), pp. 677–678.

In the first note, Kaczynski proposes the following problem, concerning group theory:

Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying: 1. K is an abelian group under addition, 2. $K - \{0\}$ is a group under multiplication, and 3. $x(y+z) = xy + xz$ for all $x, y, z \in K$. Suppose that for some n , $0 = 1 + 1 + 1 \dots + 1$ (n times). Prove that, for all $x \in K$, $(-1)x = -x$.

In the second note, the solution to the problem is — somewhat dismissively — provided by R. G. Bilyeu:

The last part of the hypothesis is unnecessary. If z denotes -1 , then $z + z + zz = z(1 + 1 + z) = z$, so $zz = 1$. Now $z(x + zx) = zx + x = x + zx$, so either $x + zx = 0$ or $z = 1$. In either case $zx = -x$.

Conclusion

Theodore J. Kaczynski was a very promising young undergraduate, graduate and post-graduate student in the 1960s. His work — although pertaining to very narrow topics — was undoubtedly, technically, first rate.

As is the case however, elegance or complexity do not themselves raise the importance of problems, achievements or for that matter, mathematicians. As expressed by his fellow graduate student Professor Peter Rosenthal in a 1996 *Toronto Star* article (after Kaczynski was charged):

[The] topic was only of interest to a very small group of mathematicians and does not appear to have broader implications; thus, his work had little impact. Kaczynski might have quit mathematics because he was discouraged by the resultant lack of recognition.

In another 1996 article, in the Los Angeles Times article, Professor Donald Rung similarly expressed:

“The field that Kaczynski worked in doesn’t really exist today [...]. He probably would have gone on to some other area if he were to stay in mathematics,” Rung said. “As you can imagine, there are not a thousand theorems to be proved about this stuff.”

An Advanced Explanation of His Breakthrough by Lara Pudwell

Original PDF: Digit Reversal Without Apology.pdf

Digit Reversal Without Apology

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In A Mathematician's Apology¹ G. H. Hardy states, "8712 and 9801 are the only four-figure numbers which are integral multiples of their reversals"; and, he further comments that "this is not a serious theorem, as it is not capable of any significant generalization."

However, Hardy's comment may have been short-sighted. In 1966, A. Sutcliffe² expanded this obscure fact about reversals. Instead of restricting his study to base 10 integers and their reversals, Sutcliffe generalized the problem to study all integer solutions of

$$k(ahn^h + a_{h-1}n^{h-1} + \dots + a_1n + a_0) = a_0n^h + a_1n^{h-1} + \dots + a_{h-1}n + a_h$$

with $n > 2$, $1 < k < n$, $0 < a_i < n - 1$ for all i , $a_0 = 0$, $a_h = 0$. We shall refer to such an integer $a_0\dots a_h$ as an $(h + 1)$ -digit solution for n and write $k(a_h, a_{h-1}, \dots, a_1, a_0)_n = (a_0, a_1, \dots, a_{h-1}, a_h)_n$. For example, 8712 and 9801 are 4-digit solutions in base $n = 10$ for $k = 4$ and $k = 9$ respectively. After characterizing all 2-digit solutions for fixed n and generating parametric solutions for higher digit solutions, Sutcliffe left the following open question: Is there any base n for which there is a 3-digit solution but no 2-digit solution?

Two years later T. J. Kaczynski⁽¹⁾³ answered Sutcliffe's question in the negative. His elegant proof showed that if there exists a 3-digit solution for n , then deleting the middle digit gives a 2-digit solution for n . Together with Sutcliffe's work, this proved that there exists a 2-digit solution for n if and only if there exists a 3-digit solution for n .

¹ F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci. U. S. A.* > 4 (1955) 379-382.

² F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

³ S. Banach, Uber analytisch darstellbare Operationen in abstrakten Raumen, *Fund. Math.*, 17 (1931) 283-295.

⁽¹⁾ Better known for other work.

Given the nice correspondence between 2- and 3-digit solutions described by Sutcliffe and Kaczynski, it is natural to ask if there exists such a correspondence for higher digit solutions. In this paper, we will explore the relationship between 4- and 5-digit solutions. Unfortunately, there is not a bijection between these solutions, but there is a nice family of 4- and 5- digit solutions which have a natural one-to-one correspondence.

A second extension of Sutcliffe and Kaczynski's results is to ask, "Is there any value of n for which there is a 5-digit solution but no 4-digit solution?" We will answer this question in the negative; and, furthermore, we will show that there exist 4- and 5-digit solutions for every $n > 3$.

An attempt at generalization

In the case of 3-digit solutions, Kaczynski proved that if $n + 1$ is prime and $k(a, b, c)n = (c, b, a)n$ is a 3-digit solution for n , then $k(a, c)n = (c, a)n$ is a 2-digit solution. Thus, we consider the following:

Question 1. Let $k(a, b, c, d, e)n = (e, d, c, b, a)n$ be a 5-digit solution for n . If $n + 1$ is prime, then is $k(a, b, d, e)n = (e, d, b, a)n$ a 4-digit solution for n ?

First, following Kaczynski, let $p = n + 1$. We have

$$k(an^4 + bn^3 + cn^2 + dn + e) = en^4 + dn^3 + cn^2 + bn + a. \quad (1)$$

Reducing this equation modulo p , we obtain

$$k(a - b + c - d + e) = e - d + c - b + a = a - b + c - d + e \pmod{p}.$$

Thus, $(k - 1)(a - b + c - d + e) = 0 \pmod{p}$, and

$$p \mid (k - 1)(a - b + c - d + e). \quad (2)$$

If $p \mid (k - 1)$, then $k - 1 > p$, which is impossible because $k < n$. Therefore, $p \mid (a - b + c - d + e)$. But $-2p < -2n < a - b + c - d + e < 3n < 3p$, so there are four possibilities:

- (i) $a - b + c - d + e = -p$,
- (ii) $a - b + c - d + e = 0$,
- (iii) $a - b + c - d + e = p$, (iv) $a - b + c - d + e = 2p$.

Write $a - b + c - d + e = fp$, where $f \in \{-1, 0, 1, 2\}$. Substituting $c = -a + b + d - e + fp$ into equation 1 gives:

$$\begin{aligned} k[n^2(n^2 - 1)a + n^2(n + 1)b + fpn^2 + n(n + 1)d - (n^2 - 1)e] \\ = n^2(n^2 - 1)e + n^2(n + 1)d + fpn^2 + n(n + 1)b - (n^2 - 1)a. \end{aligned}$$

After substituting for p , dividing by $n + 1$, and rearranging, one sees that $k[an^3 + (b - a + f)n^2 + (d - e)n + e] = en^3 + (d - e + f)n^2 + (b - a)n + a$. Indeed, this is a 4-digit solution for n if $f = 0$, $b - a > 0$, and $d - e > 0$, but not necessarily a 4-digit solution of the form conjectured in Question 1.

As in Kaczynski's proof for 2- and 3-digit solutions, it would be ideal if three of the four possible values for f lead to contradictions and the fourth leads to a "nice" pairing of 4- and 5-digit solutions. Unlike Kaczynski, we now have the added advantage of exploring these cases with computer programs such as Maple. Experimental evidence

suggests that the cases $f = -1$ and $f = 2$ are impossible. The cases $f = 0$ and $f = 1$ are discussed below.

A counterexample

Unfortunately, Kaczynski's proof does not completely generalize to higher digit solutions. Most 5-digit solutions do, in fact, yield 4-digit solutions in the manner described in Question 1, but for sufficiently large n there are examples where $(a, b, c, d, e)_n$ is a 5-digit solution but $(a, b, d, e)_n$ is not a 4-digit solution.

A computer search shows that the smallest such counterexamples appear when $n = 22$:

$$7(2, 8, 3, 13, 16)_{22} = (16, 13, 3, 8, 2)_{22}, 3(2, 16, 11, 5, 8)_{22} = (8, 5, 11, 16, 2)_{22}.$$

However, there is no integer k for which $k(2, 8, 13, 16)_{22} = (16, 13, 8, 2)_{22}$ or $k(2, 16, 5, 8)_{22} = (8, 5, 16, 2)_{22}$. Note that $-2 + 8 + 13 - 16 = 3$ and $-2 + 16 + 5 - 8 = 11$; that is, both of these counterexamples to Question 1 occur when $f = 0$. The next smallest counterexamples are

$$3(3, 22, 15, 7, 11)_{30} = (11, 7, 15, 22, 3)_{30}, 8(2, 13, 8, 16, 9)_{30} = (9, 16, 8, 13, 2)_{30},$$

which occur when $f = 0$ and $n = 30$.

A family of 4- and 5-digit solutions

Although Kaczynski's proof does not generalize entirely, there exists a family of 5-digit solutions when $f = 1$ that has a nice structure.

Theorem 1. Fix $n > 2$ and $a > 0$. Then

$$k(a, a - 1, n - 1, n - a - 1, n - a)_n = (n - a, n - a - 1, n - 1, a - 1, a)_n$$

is a 5-digit solution for n if and only if $a \mid (n - a)$.

Proof. We have

$$(n - a)n^4 + (n - a - 1)n^3 + (n - 1)n^2 + (a - 1)n + a \quad an^4 + (a - 1)n^3 + (n - 1)n^2 + (n - a - 1)n + (n - a) \\ (n - a)(n^4 + n^3 - n - 1) \quad n - a \quad a(n^4 + n^3 - n - 1) \quad a,$$

and the result is clear. \square

Notice that

$$(-a + (a - 1)) + ((n - a - 1) - (n - a)) + p = -1 + -1 + (n + 1) = n - 1.$$

That is, this family of solutions occurs when $f = 1$. Moreover, this family follows the pattern described in Question 1; that is, for each 5-digit solution described in Theorem 1, deleting its middle digit gives a 4-digit solution.

Theorem 2. If

$$k(a, a - 1, n - 1, n - a - 1, n - a)_n = (n - a, n - a - 1, n - 1, a - 1, a)_n$$

is a 5-digit solution for n , then

$$k(a, a - 1, n - a - 1, n - a)_n = (n - a, n - a - 1, a - 1, a)_n$$

is a 4-digit solution for n .

Proof. By Theorem 1, $n - a \mid G(N)$. Now

$$\begin{aligned} & (n - a)n^3 + (n - a - 1)n^2 + (a - 1)n + a \\ &= an^3 + (a - 1)n^2 + (n - a - 1)n + (n - a) \\ &= (n - a)(n^3 + n^2 - n - 1) + n - a \\ &= a(n^3 + n^2 - n - 1) + a. \end{aligned}$$

□

These 4-digit solutions were first described by Klosinski and Smolarski⁴ in 1969, but their relationship to 5-digit solutions was not made explicit before now.

It is also interesting to note that 9801 and 8712, the two integers in Hardy's discussion of reversals, are included in this family of solutions.

We conclude with the following corollary.

Corollary 1. There is a 4-digit solution and a 5-digit solution for every $n > 3$.

Proof. Let $a = 1$ in the statements of Theorem 1 and Theorem 2 above. □

Some open questions

We have shown that there is no n for which there is a 5-digit solution but no 4-digit solution. More specifically, we know that there are 4- and 5-digit solutions for every $n > 3$.

Although Kaczynski's proof does not generalize directly to 4- and 5-digit solutions, it does bring to light several questions about the structure of solutions to the digit reversal problem.

First, it would be interesting to completely characterize 4- and 5-digit solutions for n . Namely,

1. All known counterexamples to Question 1 occur when $f = 0$. Are there counterexamples for which $f \neq 0$? Is there a parameterization for all such counterexamples?

2. Theorems 1 and 2 exhibit a family of 4- and 5-digit solutions for $f = 1$ with a particularly nice structure. To date, no other 4- or 5-digit solutions are known for $f = 1$. Do such solutions exist?

More generally,

3. Solutions to the digit reversal problem have not been explicitly characterized for more than 5 digits. Do there exist analogous results to Theorems 1 and 2 for higher digit solutions?

A Maple package for exploring these questions is available from the author's web page at <http://www.math.rutgers.edu/~lpudwell/maple.html>.

⁴ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

Acknowledgment

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References

Ted's Work as a Michigan PhD Student

1. June 1964 - Another Proof of Wedderburn's Theorem

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Another Proof of Wedderburn's Theorem

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ANOTHER PROOF OF WEDDERBURN'S THEOREM

T. J. Kaczynski, Evergreen Park, Illinois

In 1905 Wedderburn proved that every finite skew field is commutative. At least seven proofs of this theorem (not counting the present one) are known. See^{1,2,3} (Part Two, p. 206 and Exercise 4 on p. 219),⁴ (two proofs), and⁵. Unlike these proofs, the

¹ F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci. U. S. A.* > 4 (1955) 379-382.

² F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

³ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

⁴ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

⁵ H. Tietze, Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.*, 5 (1919), 284-291.

proof to be given here is group-theoretic, in the sense that the only non-group-theoretic concepts employed are of an elementary nature.

Lemma. Let q be a prime. Then the congruence $Z^2 + r^2 \equiv -1 \pmod{q}$ has a solution t, r with $t \not\equiv 0 \pmod{q}$.

Proof. If -1 is a quadratic residue, take $r = 0$ and choose t appropriately. Assume -1 is a nonresidue. Then any nonresidue can be written in the form $-s^2 \pmod{q}$ with $s \not\equiv 0$. If $t^2 + r^2$ is ever a nonresidue for some t, r , set $t^2 + r^2 \equiv -s^2$, and we have $(t/s)^2 + (r/s)^2 \equiv -1$. (Throughout this note, x^{-1} denotes that integer for which $xx^{-1} \equiv 1 \pmod{q}$.) On the other hand, if $t^2 + r^2$ is always a residue, then the sum of any two residues is a residue, so $-1 \equiv g^{-1} \equiv 1 + 14 \dots + 1$ is a residue, contradicting our assumption.

Proof of the theorem. Let F be our finite skew field, E^* its multiplicative group. Let 5 be any Sylow subgroup of F^* , of order, say, p^a . Choose an element g of order p in the center of 5 . If some $h \notin 5$ generates a subgroup of order p different from that generated by g , then g and h generate a commutative field containing more than p roots of the equation $x^p = 1$, an impossibility. Thus 5 contains only one subgroup of order p and hence is either a cyclic or a generalized quaternion group (⁶ p. 189).

If 5 is a generalized quaternion group, then 5 contains a quaternion subgroup generated by two elements a and b , both of order 4, where $ba = -a^{-1}b$. Now a^2 generates a commutative field in which the only roots of the equation $x^2 = 1$ or $(x+1)(x-1) = 0$ are ± 1 , so since $(a^2)^2 = 1$, we have

$$(1) \ a^2 = -1.$$

Hence $a^{-1} = a^2 = -a$ so

$$(2) \ ba = -ab.$$

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Similarly,

$$(3) \ 5^2 = -1.$$

Taking $q = \text{characteristic of } F$ ($\neq 1 = 0$), choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

$$(t^2 + ra + 5)(r^2 + 1 + rta + tb) = r(t^2 + r^2 + 1)a + (t^2 + r^2 + 1)5 = 0.$$

One of the factors on the left must be 0, so for some numbers u, v, w , $u \not\equiv 0 \pmod{q}$, we have $w + a + 5 = 0$, or $b = -u^{-1}wa - u^{-1}w$. So b commutes with a , a contradiction. We conclude that 5 is not a generalized quaternion group, so 5 is cyclic.

Thus every Sylow subgroup of F^* is cyclic, and F^* is solvable (⁷, pp. 181—182). Let Z be the center of F^* and assume $Z \neq F^*$. Then F^*/Z is solvable, and its Sylow subgroups are cyclic. Let A/Z (with $ZC \leq 4$) be a minimal normal subgroup of F^*/Z . A/Z is an elementary abelian group of order $(F \text{ prime})$, so since the Sylow subgroups of

⁶ S. Banach, *Über analytisch darstellbare Operationen in abstrakten Räumen*, *Fund. Math.*, 17 (1931) 283-295.

⁷ P. T. Church, *Ambiguous points of a function homeomorphic inside a sphere*, *Michigan Math. J.*, 4 (1957) 155-156.

F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be any element of F^* , y any element of A . Since A is normal, $xyx^{-1} \in A$, and $(1+x)y = z(1+x)$ for some $z \in A$. An easy manipulation shows that $y - z - zx - xy = (z - xyx^{-1})x$.

If $y - z = z - xyx^{-1} = 0$, then $y = z = xyx^{-1}$, so x and y commute. Otherwise, $x = (z - xyx^{-1})^{-1}(y - z)$. But A is abelian, and $2, y, xyx^{-1} \in A$, so x commutes with y . Thus we have proven that A is contained in the center of F^* , a contradiction.

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A NOTE ON PRODUCT SYSTEMS OF SETS OF NATURAL NUMBERS

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In this note, we apply a slight twist to a trick exploited about twelve years ago by J. C. E. Dekker ([2]), our purpose being to expose a couple of elementary facts about nonempty, countable "product systems" of infinite sets of natural numbers which are, at the same time, "finite symmetric difference systems." We proceed in terms of the following definitions.

Definition. By a product system of subsets of N (N the natural numbers), we mean a collection of subsets of N which contains, along with any two of its members, their intersection.

2. 1964 - Distributivity and $(-1)x = -x$ (Advanced Problem 5210)

Original PDF: 2. 1964 Distributivity and $(-1)x = -x$ (Advanced Problem 5210).pdf

Kaczynski, T.J. (1964). "Distributivity and $(-1)x = -x$ (Advanced Problem 5210)". *The American Mathematical Monthly*. 71(6), pp. 689.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers - The State University, New Brunswick, N.J. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before December 31, 1964

5210. *Proposed by T. J. Kaczynski, Evergreen Park, Illinois*

Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying:

1. K is an abelian group under addition,
2. $K - \{0\}$ is a group under multiplication, and
3. $x(y + z) = xy + xz$ for all $x, y, z \in K$.

Suppose that for some n , $0 = 1 + 1 + \dots + 1$ (n times). Prove that, for all $x \in K$, $(-1)x = -x$.

3. 1964 - Distributivity and $(-1)x = -x$ (Advanced Problem 5210, with Solution by Bilyeu, R.G.)

Original PDF: 3. Distributivity and $(-1)x = -x$ (Advanced Problem 5210, with Solution by Bilyeu, R.G.).pdf

Kaczynski, T.J. (1965). “Distributivity and $(-1)x = -x$ (Advanced Problem 5210, with Solution by Bilyeu, R.G.)”. *The American Mathematical Monthly* 72(6), pp. 677–678.

Distributivity and $(-1)x = -x$

5210 [1964, 689]. *Proposed by T. J. Kaczynski, Evergreen Park, Illinois*

Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying:

1. K is an abelian group under addition,
2. $K - \{0\}$ is a group under multiplication, and
3. $x(y + z) = xy + xz$ for all $x, y, z \in K$.

Suppose that for some n , $0 = 1 + 1 + \dots + 1$ (n times). Prove that, for all $x \in K$, $(-1)x = -x$.

Solution by R. G. Bilyeu, North Texas State University. The last part of the hypothesis is unnecessary. If z denotes -1 , then $z + z + z = (1 + 1 + 1)z = z$, so $z(3) = 1$. Now $z(x + zx) = zx + x = x + zx$, so either $x + zx = 0$ or $z = 1$. In either case $z(x) = -x$.

Also solved by Carol Avelsgaard, Richard Bourgin, Robert Bowen, Joel Brawley, Jr., F. P. Callahan, M. M. Chawla (India), R. A. Cunningham-Green (England), M. J. DeLeon, M. Edelstein, N. J. Fine, Harvey Friedman, Anton Glaser, M. G. Greening (Australia), A. G. Heinicke, Sidney Heller, G. A. Heuer, Stephen Hoffman, K. G. Johnson, A. J. Karson, Max Klicker, Kwangil Koh, C. C. Lindner, C. R. MacCluer, H. F. Mattson, C. J. Maxson, R. V. Moddy, Jose Morgado (Brazil), W. L. Owen, Jr., P. R. Parthasarathy (India), Harsh Pittie, Kenneth Rogers, Toru Saito (Japan), Camilio Schmidt, Leonard Shapiro, Frank A. Smith, George Van Zwahlenberg, W. C. Waterhouse, Kenneth Yanosko, and the proposer.

4. 1965 - Boundary Functions for Functions Defined in a Disk

Original PDF: 4. 1965 Boundary Functions for Functions Defined in a Disk.pdf

Kaczynski, T.J. 1965. Boundary functions for functions defined in a disk. *J. Math. and Mech.* 14(4):589-612.

MR0176080 Kaczynski, T. J. Boundary functions for function defined in a disk. *J. Math. Mech.* 14 1965 589.612. (Reviewer: C. Tanaka) 30.62

Explanation by John D. Bullough

Let D denote the unit disk $|z| < 1$, C its boundary, and let $f(z)$ be any function that is defined in D and takes its values in some metric space S . Then a boundary function for f is a function t on C such that for every $x \in C$ there exists an arc v at x with

$$\lim_{z \rightarrow x} f(z) = t(x).$$

$$z \rightarrow x$$

$$z \in v$$

The author proves several theorems on boundary functions in the following four cases: (1) $f(z)$ a homeomorphism of D onto D , (2) $f(z)$ a continuous function, (3) $f(z)$ a Baire function and (4) $f(z)$ a measurable function. These theorems include answers to two questions raised by Bagemihl and Piranian.

Theorem 1 states that if $f(z)$ is a homeomorphism of D onto D , then there exists a countable set N such that $t|_{C - N}$ is continuous.

In the case of continuous functions, one needs some definitions. Let S and T be metric spaces. f is said to be of Baire class 1(S, T) if and only if (i) domain $f = S$, (ii) range $f \subset T$ and (iii) there exists a sequence $\{f(n)\}$ of continuous functions, each mapping S into T , such that $f(n) \rightarrow f$ pointwise on S . g is of honorary Baire class 2(S, T) if and only if (i) domain $g = S$, (ii) range $g \subset T$ and (iii) there exists a function f of Baire class 1(S, T) and a countable set N such that $f|_{S - N} = g|_{S - N}$. Using these definitions, Theorems 2 and 3 read as follows. Theorem 2: Let f be a continuous real-valued function in D and let t be a finite-valued boundary function for f . Then t is of honorary Baire class 2(C, R), where R is the set of real numbers. Theorem 3:

Let f be a continuous function mapping D into the Riemann sphere S and let t be a boundary function for f . Then t is of honorary Baire class $2(C, S)$.

In the cases of Baire functions and measurable functions, for the sake of convenience consider the open upper half-plane $D^0: I(z) > 0$, and its boundary $C^0: I(z) = 0$, instead of D and C , respectively. Theorem 4 states that if f is a real-valued function of Baire class $\alpha > 1$ in D^0 , and t is a finite-valued boundary function, then t is of Baire class $\alpha + 1$. As an immediate consequence of Theorem 4, one has Theorem 5: Let f be a real-valued Borel-measurable function in D^0 and let t be a finite-valued boundary function for f ; then t is Borel-measurable.

Next, the author proves that for an arbitrary function t on C^0 , there exists a function f on D^0 such that $f(z) = 0$ almost everywhere and t is a boundary function for f . The paper concludes with some remarks concerning extensions of these theorems into three dimensions.

Article by Ted

Boundary Functions for Functions Defined in a Disk

T. J. KACZYNSKI

Communicated by F. Bagemihl

1. Introduction

Throughout this paper D will denote the open unit disk (in two-dimensional Euclidean space) and C will denote its boundary, the unit circle. Bagemihl and Piranian¹ have introduced the following definition.

Definition. If $x \in C$, an arc at x is \mathcal{E} simple arc y having one endpoint at x such that $y \cap \{x\} = \{x\} \cap C = D$. Let f be any function that is defined in D and takes its values in some metric space S . Then a *boundary junction* for f is a function ϕ on C such that for every $x \in C$ there exists an arc y at x with

$$\lim_{z \rightarrow x} f(z) = \phi(x).$$

The purpose of this paper is to prove several theorems concerning boundary functions. These theorems include answers to two questions raised in² (see Problem 1 and the conjecture on p. 202).

The set of real numbers will be denoted by R , N -dimensional Euclidean space will be denoted by R^N , and the Riemann sphere will be denoted by Σ . Points in R^N will be written in the form $\{x_1, x_2, \dots, x_N\}$ rather than (x_1, x_2, \dots, x_N) (to avoid confusion with open intervals of real numbers in the case $N = 2$). Whenever

¹ F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

² F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

we speak of real-valued functions we mean finite-valued functions, and whenever we speak of increasing functions we refer to weakly increasing (nondecreasing) functions. The abbreviations “l.u.b.” and “g.l.b.” stand for “least upper bound” and “greatest lower bound” respectively. Finally, it should be noted that our definition of the Baire classes is slightly unconventional (see p. 6 and p.14) in that we consider Baire class a to include Baire class ft for every $ft < a$.

2. Boundary functions for homeomorphisms.

Definition. If $E \subset D$, let $\text{acc}(E)$ denote the set of all points on C which are accessible by arcs in E .

¹I would like to thank Professor G. Piranian for his encouragement.

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Journal of Mathematics and Mechanics, Vol. 14, No. 4 (1965).

Lemma 1. Let A be an arcwise connected subset of D and let B be a connected subset of D . Suppose that $A \cap B \neq \emptyset$. Then $\text{acc}(A)$ and B have at most two points in common.

Proof. Assume that p_1, p_2, p_3 are three distinct points of $\text{acc}(A) \cap B$ and derive a contradiction. Let γ_i be an arc joining p_i to a point $q_i \in A$, with $\{p_i\} \cap A = \emptyset$ ($i = 1, 2, 3$). Let γ be an arc in A joining p_1 and p_2 . Putting

γ_1, γ_2 and γ together, we obtain an arc T joining p_1 to p_2 , with $r \in \{p_1, p_2\} \cap C \cap A$. We can assume T is a simple arc, for if T is not simple, and p_2 can be joined by some simple arc $F \subset T$ (see³). Let L_1, L_2 be the two open arcs of C determined by the pair of points p_1, p_2 . We may assume, by symmetry, that $p_3 \in L_1$. According to⁴ (Theorem 11.8, p. 119), $D - T$ has two components U_1 and U_2 , the boundary of U_1 being $L_1 \cup T$ and the boundary of U_2 being $L_2 \cup T$.

Let γ' be an arc in A joining p_3 to a point $q \in T \cap A$. Putting γ_3 and γ' together, we obtain an arc δ joining p_3 to q . Starting at p_3 and proceeding along δ , let r be the first point of T that we reach. Let A be the subarc of δ with endpoints at p_3 and r . Clearly, $A \cap \{p_3\} \cap C \cap A$. We can assume (according to⁵) that A is a simple arc.

Since $p_3 \in L_1$, p_3 is not in U_2 . Since

$A \cap \{p_3, r\} \subset D - T = U_1 \cup U_2$,

$A \cap \{p_3, r\}$ must have a point in U_1 . But $A \cap \{p_3, r\}$ is connected, so $A \cap \{p_3, r\} \subset U_1$. Hence A is a cross cut of U_1 . Let M_r, M_2 be the two open subarcs of L_1 with endpoints p_1, p_3 and p_2, p_3 respectively. Let P_1, P_2 be the two closed subarcs of T

³ H. Tietze, Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.*, 5 (1919), 284-291.

⁴ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

⁵ H. Tietze, Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven, *Math. Z.*, 5 (1919), 284-291.

with endpoints p_r, r and p_2, r respectively. According to⁶ (Theorem 11.8, p. 119), U_r — A has two components V_i and V_2 , the boundary of V_i being $kJ r_x kJ A$ and the boundary of V_2 being $M_2 kJ r_2 kJ A$.

Since $P \cup A \subset 2I$, $V_i \cap J V_2 \cap J U_2$. Recall that $p_3 \notin U_2$. It follows that since $p_3 \in B$, B has a point in common with $V_i \cap J V_2$. But B is connected, so $B \not\subset V_i \cap J V_2$. We see that $p_r \notin V_2$, and therefore that $B \cap V_i \neq \emptyset$ (because $P \in B$). Hence $B \not\subset V_i$, so $p_2 \in W$. But, since the boundary of V_i is M_x

$I \setminus kJ A, p_2 \in V_i$. This contradiction proves the lemma.

Lemma 2. There exists a countable family \mathcal{S} of open disks such that every open set $U \subset \mathbb{R}^2$ can be written in the form $U = \bigcup S_n$, where $S_n \in \mathcal{S}$ and $S_n \cap U = \emptyset$.

Proof. Let $\{p_n\}$ be a countable dense subset of \mathbb{R}^2 , and let \mathcal{S} be the family of all open disks of rational radius having some p_n as center. \mathcal{S} is clearly countable. If U is an open set it is easy to show that for each $x \in U$ there exists an $S_x \in \mathcal{S}$ with $x \in S_x \subset U$. Obviously

$$U = \bigcup S_x.$$

xt.U

Theorem 1. Let f be a homeomorphism of D onto D , and let $\langle p \rangle$ be a boundary function for f . Then there exists a countable set N such that $\langle p \rangle|_{C \setminus N}$ is continuous.

Proof. Take an arbitrary $a \in S$. It is easily shown that $D \setminus S$ and $D \cap S$ are both connected, so $f^{-1}(D \setminus S)$ and $f^{-1}(D \cap S)$ are both connected. Given $x_0 \in C$, let y be any arc at x_0 . If

$$x_0 \in \text{acc}(f^{-1}(D \cap S)),$$

then we can choose points on y arbitrarily close to x_0 which are not in $f^{-1}(D \setminus S)$, so

$$x_0 \in D \cap \text{acc}(f^{-1}(D \cap S)) = D \cap \text{acc}(D \cap S).$$

This shows that

$$(1) \quad C \cap \text{acc}(D \cap S) \subset \text{acc}(D \cap S).$$

Let

$$F = \text{acc}(f^{-1}(D \cap S)) \cap f^{-1}(D \cap S).$$

By Lemma 1, F contains at most two points, and from (1) we see that $\text{acc}(f^{-1}(D \cap S)) = F \cup (C \cap \text{acc}(D \cap S))$.

Thus we have shown that for each $a \in S$ we can write

$$\text{acc}(f^{-1}(D \cap S)) = F \cup \bigcup G_s,$$

where F_s is finite and G_s is open (relative to C).

For any arc y at a point x on C , the *cluster set* $C(f, t)$ of f along y is defined by

$C(f, t) = \{w \in \mathbb{R}^2 \mid \text{there exists a sequence } \{z_n\} \subset y \text{ such that } z_n \rightarrow w \text{ and } f(z_n) \rightarrow t\}$.

Let

$$E = \{x \in C \mid \text{there exist arcs } y_1, y_2 \text{ at } x \text{ such that } C(f, y_1) \cap C(f, y_2) = \emptyset\}.$$

⁶ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

A theorem of Bagemihl⁷ states that E is countable. Let $N = E \setminus J F_s$.

Step 1

N is countable. Let $\langle p \rangle$ denote the restriction of $\langle p \rangle$ to $C \setminus N$.

If U is any open set, write $U = S_n$, where $S_n \in \mathcal{S}$, $S_n \subset Z \cap U$. Suppose $x \in (pQ^1(U))$. Then $\langle x \rangle = \langle p(x) \rangle \in S_n$ for some n , which implies that $a; \in \text{acc}(\bigcup_{n=1}^{\infty} (S_n \setminus C \setminus D))$. Thus $\forall \epsilon > 0 \exists W \subset C \setminus J \text{acc}(f \setminus S_n \cap Z) \cap N$.

On the other hand, suppose $x \in \text{acc}(IP D)$ for some n , and $x \notin N$. Choose an arc y in $fX S_n \setminus (P \setminus D)$ with one endpoint at x . Clearly,

$C(f, y) \subset S_n \setminus (P \setminus D) \subset S_n \setminus CU$.

Since $x \notin E$,

$\langle \cdot \rangle = p(\cdot) \in C(j, y) \subset U$,

so $x \in \bigcup_{n=1}^{\infty} (G_n)$. Thus

$\setminus J \text{acc}(T^1(S_n \cap D) \setminus NC \cap U), n$

Step 2

$\langle Po^1(U) \rangle = U \cap \text{acc}(r \setminus S_n, f \setminus D) \setminus N = \setminus J (F_s, U \setminus G_{s,n}) \setminus N$

$= \bigcup_{s=1}^{\infty} (U \cap S_s) \cap (C \setminus N)$.

Thus, for each open set U , $\langle Po^1(U) \rangle$ is an open set relative to $C \setminus N$. Therefore $\langle p \rangle$ is continuous. *Q.E.D.*

3. Boundary functions for continuous functions.

Definition. Let \mathcal{S} and T be metric spaces. We will say the function f is of *Baire class 1* (\mathcal{S}, T) if, and only if,

- (i) domain $f = \mathcal{S}$,
- (ii) range $f \subset T$, and
- (iii) there exists a sequence $\{f_n\}$ of continuous functions, each mapping \mathcal{S} into T , such that $f_n \rightarrow f$ pointwise on \mathcal{S} .

We will say the function g is of *honorary Baire class 2* (\mathcal{S}, T) if, and only if, (i) domain $g = \mathcal{S}$, (ii) range $g \subset T$, and

- (iii) there exists a function f of Baire class 1 (\mathcal{S}, T) and a countable set N such that $f \setminus N = g \setminus N$.

Lemma 3. Let f be a continuous real-valued function in D and let $\langle p \rangle$ be a finitevalued boundary function for f . Let r and t be real numbers with $r < t$. Then

(A) there exists a G_δ set G and a countable set N such that

$\bigcap_{n=1}^{\infty} (f_n, 4 - \epsilon) \subset G \subset \bigcap_{n=1}^{\infty} (f_n, +\infty) \setminus N$, and

(B) there exists a G_δ set H and a countable set M such that

$t \in H \subset (-\infty, r] \setminus M$.

Proof. Let

⁷ F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci. U. S. A.* 4 (1955) 379-382.

$t - r^e \sim 2$,
 $C_n = Lit^1 \mid |z| = 1 - 4, (n)$
 $A_n = \set{ztR^2 \mid 1 - - < > \mid < 4, (nJ}$
 $E_n = \{x \in C \mid \text{there exists an arc } y \text{ at } x \text{ having one endpoint on } C_n, \text{ with } y = \{x\}$
 $Q f^1((-^{00}, r))$,
 $K = \{x \in C \mid \text{there exists an arc } y \text{ at } x \text{ with } y = \{\$ \} c f^1 \wedge - c, + \langle \rangle)\}$.
 Observe that
 $\wedge((-co.r)) C \set{jE_n, n \ll 1}$
 and
 $< \set{ / - e, + \langle \rangle } C K$.

For the time being, let n be a fixed integer. If $x \mathcal{L} K$, we can find an arc y_g at x such that

$$y^* - M C A_n C \set{f \mid |t - c, + \langle \rangle}).$$

Since an arc at x is by definition a simple arc, $y_x = \{x\}$ is a connected set. It follows that $y_x = \{rc\}$ must be contained entirely within one component of the open set $A_{nn}(/-e, +-)$.

We denote this component by U_x . U_x is a nonempty open connected set. Let T be the set of all points of K which are two-sided limit points of E_n .

Assertion. If $x, y \mathcal{L} T$ and $x \neq y$, then $U_x \cap U_y = \langle j \rangle$.

To prove this assertion we assume that z is a point of $U_x \cap U_y$ and we derive a contradiction. Choose points x' and y' in $y_x = \{x\}$ and $y_y = \{y\}$ respectively. Join x to x' by an appropriate subarc of y_x . Join x' to z by an arc in U_x . Join z to y^* by an arc in U_y . Join y' to y by a subarc of y_y . Putting these arcs together, we obtain an arc a with endpoints at x and y such that

$$a = \{x, y\} C A_n C \set{f \sim (t - e, + co)}.$$

We can assume that a is a simple arc, for if a is not a simple arc we can replace a by a simple arc $a' \subset a$ having endpoints at x and y (see⁸). a is a crosscut of D . Let L_1 and L_2 be the two open arcs of C determined by x and y . According to⁹ (Theorem 11.8, p. 119), $D - a$ has two components, P_1 and V_2 , whose boundaries are $L_1 \cup a$ and $L_2 \cup a$ respectively. From the fact that C_n is connected and does not intersect a it follows that C_n is contained entirely within one component of $D - a$. By symmetry, we may assume $C_n \mathcal{L} V_2$.

Since $\$$ is a two-sided limit point of E_n , Lr must contain a point of E_n , and hence a point of E_n . Say $w \mathcal{L} A E_n$. There exists a simple arc ft joining w to some point on C_n , with

$$ft = \{w\} C^{00}, r).$$

$ft = \{w\}$ cannot have a point in common with a , because

⁸ H. Tietze, *Über stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, *Math. Z.*, 5 (1919), 284-291.

⁹ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

$a \sim \{x, y\} \subset f^{-1}(t - \epsilon, +^{00})$, and
 $e, +^{-}) = \langle / \rangle$.

Thus $C_n \cup (0 - \{w\})$ is a connected set not meeting a . $C_n \cup (/3 - \{w\})$ meets y_2 , so $C_n \cap V_2 \cap (0 - \{w\}) \subset V_2$. Consequently, w is in the boundary of V_2 . But this is a contradiction, because $w \in L_x$ and the boundary of V_2 is $L_2 \cup a$. This proves the assertion.

From the assertion it follows immediately that T is countable; for any family of disjoint nonempty open sets is countable. We know that the set S of all points of E_n which are not two-sided limit points of E_n is countable.

$$K \cap C \cap E_n = [K \cap C \cap S] \cup [K \cap (E_n - S)] = (K \cap C \cap S) \cap T.$$

This shows that (for any r) $K \cap C \cap E_n$ is countable. So if we let

$$N = K \cap C \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (K \cap C \cap E_n),$$

then N is a countable set. Let

$$G = C - \bigcup_{n=1}^{\infty} E_n.$$

G is a G_δ set. Using the fact that

$$\langle^* ((-\infty, r)) \cap \bigcup_{n=1}^{\infty} E_n \cap \bigcup_{n=1}^{\infty} E_n, n=1, n=1$$

we find that

00

$$C - \langle^* ((-\infty, r)) \cap \bigcup_{n=1}^{\infty} E_n = G \cap K - N.$$

$n=1$

But

$$= \wedge([r, +^{-}))$$

and

$$K \cap \wedge((/ - \epsilon, +\langle \rangle) \cap \langle^* ((-\infty, r)) \cap \bigcup_{n=1}^{\infty} E_n),$$

$$\langle X[r, +\infty) \rangle - N.$$

This proves (A). To prove (B), simply replace $/$ and by $-f$ and and apply (A).

Theorem 2. Let f be a continuous real-valued function in D , and let $\langle p$ be a finite-valued boundary function for f . Then f is of honorary Baire class 2(C, R).

Proof. For each pair of rational numbers r and t with $r < t$, choose G_δ sets $G(r, t)$, $H(r, t)$ and countable sets $N(r, i)$, $M(r, f)$ such that

$$+\langle \rangle) \cap G(r, t) \cap \langle^* ((-\infty, r)) \cap \bigcup_{n=1}^{\infty} E_n = N(r, f),$$

$$\langle^* ((-\infty, t)) \cap H(r, f) \cap \langle^* ((-\infty, r)) \cap \bigcup_{n=1}^{\infty} E_n = M(r, Z).$$

Let

$$N = \bigcup \{N(r, t) \cup M(r, f)\},$$

where the union is taken over all pairs of rationals r, t with $r < t$. N is countable.

Let $(p_0$ denote the restriction of f to $C - N$, and let $G^*(r, t) = G(r, t) - N$. Since every countable set is an F_σ set, $G^*(r, t)$ is a G_δ set. Observe that

$$(2) \wedge([r, +^{-})) = \wedge([r, +\langle \rangle) \cap N \cap G^*(r, f)$$

$$\cap \langle^* ((-\infty, r)) \cap \bigcup_{n=1}^{\infty} E_n = \wedge([Z, +^{-})) \bullet$$

If t is a fixed rational number, let $\{r_n\}$ be a strictly increasing sequence of rational numbers converging to t . Then, by (2),

$$C \cap \langle^* ((-\infty, r_n)) \cap \bigcup_{n=1}^{\infty} E_n \cap G^*(r_n, f) \cap \langle^* ((-\infty, t)) \cap \bigcup_{n=1}^{\infty} E_n = \wedge([r_n, +\langle \rangle),$$

$n=1 \quad n=1 \quad n=1$

SO

$\wedge'(R, +^{\circ\circ}) = C \setminus G^*(r_n, f). \quad n = 1$

This proves that for every rational t , $\wedge'([J, +^{\circ\circ}])$ is a G_δ set.

If u is any real number, choose a strictly increasing sequence $\{\mathcal{L}_n J$ of rational numbers converging to u . Then

$+^{\circ\circ}) = fW_oW_n, +^{\circ\circ}), \quad n = 1$

SO $\langle p-Q \setminus [u, +^{\circ\circ}) \rangle$ is a G_δ set. By a similar argument, we find that $\wedge^{\circ 1}((\text{---}^{\circ\circ}, u/)$ is a G_δ set for every real u . So

$\rangle, +^-) = (C - N) \cap (C - , u/y)$

is the intersection of an F_{σ} set with $C - N$. By a theorem stated on p. 309 of Hausdorff's paper¹⁰, $\langle p_0$ can be extended to a real-valued function on C such that for every real u , $+^{\circ\circ})$ is a G_δ set and $+ \langle \rangle$ is an F_σ set. By Theorem IX of the same paper, $\langle p$ is of Baire class 1(C , \mathbb{R}). Since $\langle p(x) = p(x)$ except for $x \in N$, $\langle p$ is of honorary Baire class 2(C , \mathbb{R}). *Q.E.D.*

Corollary. Let f be a continuous \mathbb{R} -junction mapping D into \mathbb{R}^N , and suppose $\langle p : C \rightarrow \mathbb{R}^N$ is a boundary junction for j . Then $\langle p$ is of honorary Baire class 2(C , \mathbb{R}^N).

Proof. We simply write our functions in terms of their components, say

$f = (f_1, f_2, \dots, f_N)$, and $\langle p = (\langle p_1, \langle p_2, \dots, \langle p_N)$.

Obviously $\langle p_i$ is a boundary function for \mathbb{R} , and so is of honorary Baire class 2(C , \mathbb{R}). We choose a function of Baire class 1(C , \mathbb{R}) which agrees with $\langle p_i$ except on a countable set M_i . Setting

$0 = \langle 0_i, 0_2, \dots, 0_N \rangle$,

it is clear that g is of Baire class 1(C , \mathbb{R}^N), and that g agrees with $\langle p$ except on the countable set $\bigcup M_i$. Hence $\langle p$ is of honorary Baire class 2(C , \mathbb{R}^N).

Q.E.D.

Lemma 4. Let g be a continuous function mapping C into \mathbb{R}^3 . Let q be a point of \mathbb{R}^3 and let ϵ be a positive real number. Then there exists a continuous function $g^* : C \rightarrow \mathbb{R}^3$ such that q does not lie in the range of g^* , and for all $x \in C$,

$|g(x) - q| < \epsilon \Rightarrow g(x) = g^*(x).$

Proof. Let

$S = \{y \in \mathbb{R}^3 \mid |y - q| < \epsilon\}.$

If $0 \in C$, let $g^* : C \rightarrow \mathbb{R}^3$ be any continuous function whose range does not include q . Otherwise, $0^{-1}(S)$ is a proper open subset of C and hence can be written in the form

$g^{-1}(S) = \bigcup A_k$,

where

$A_k = \{e^{it} \mid a_k < t < b_k\}$, and

$a_k < b_k$.

¹⁰ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

Since $0^{-1}(\{0\})$ is a closed (and therefore compact) subset of $0^{-1}(]S)$, $0^{-1}(\{^{\wedge}\})$ is covered by a finite number of I_{fc} 's. Say

$$0^{-1}(\{0\}) \dots \setminus JI_n.$$

The endpoints e^{iak} and e^{t6A} of I_k are not in $0^{-1}(\{<?\})$, so we can construct, for each k , a continuous function $g_k : I_k \rightarrow R^3$ such that

$$Sk(e^{iai}) = g(e^{iai}), = g^{\wedge*},$$

and q is not in the range of g_k . Define

$$0^*(x) = 0(a:), \text{ if } 0;_x C^{\sim} (AUZ_2U \dots U/n),$$

$$0^*(x) = g_k(x), \text{ if } x \in I_k, k = 1, \bullet \bullet \bullet, n.$$

It is easy to show that g^* has the desired properties.

Theorem 3. Let f be a continuous function mapping D into the Riemann sphere 2, and let $<p$ be a boundary function for f . Then $<p$ is of honorary Baire class 2($C, 2$).

Proof. Since 2 is a subset of $J^{?3}$, the corollary to Theorem 2 shows that $<p$ is of honorary Baire class 2($(7, 2^{?3})$). Let g be a function of Baire class 1($C, 2^{?3}$) which differs from $<p$ only on a countable set N . Then $g(C) - 2$ is countable, so there exists a point q inside of 2 (that is, in the bounded open domain determined by 2) which is not in the range of g . Let $\{</,\}$ be a sequence of continuous functions converging to g . By Lemma 4 we can find (for each ri) a continuous function $g_n^* : C \rightarrow R^3$ such that q does not lie in the range of g^* , and for all $x \notin C$,

$$-g|g_n(x) = g^*(x). a$$

It is easy to show that $g_n^* \rightarrow g$.

We define a function P as follows. If $a \in R^3 - \{q\}$, let I be the unique ray with endpoint at q that passes through a , and let $P(a)$ be the intersection point of I with 2. Obviously, P is a continuous mapping of $R^3 - \{<?\}$ onto 2, and P fixes every point of 2. Therefore

$$F(0(z)) = \text{if } x^{\wedge}N,$$

$$P(g_n^*(x)) \text{ is a continuous function from } C \text{ into } 2, \text{ and}$$

$$F0(x)) \text{ as } n \rightarrow \infty.$$

This shows that $<p$ is of honorary Baire class 2($C, 2$). *Q.E.D.*

4. Boundary functions for Baire functions.

In this section we concern ourselves only with real-valued functions. We shall prove that a boundary function for a function of Baire class $a + 1$ is of Baire class $a + 1$. It is convenient to prove this theorem for functions that are defined in the (open) upper halfplane and have boundary functions defined on the rr -axis rather than for functions defined in D . Once the theorem is proved in this form it is a routine computational matter to show that it also holds for functions defined in D . The reader should be familiar with the results of Hausdorff¹¹ before reading this section. Unfortunately, we must begin with some tedious preliminaries.

¹¹ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-

Let

We will regard C° as being identical with R .

Suppose \mathcal{S} is a metric space. Let \mathcal{G} be the class of all open sets of \mathcal{S} and let \mathcal{C} be the class of all closed sets of \mathcal{S} .

A function $f : \mathcal{S} \rightarrow R$ is of *Baire class 0* if and only if it is continuous. For any ordinal number $\alpha > 0$, f is of *Baire class α* if and only if f is the pointwise limit of a sequence of functions each of Baire class less than α .

Let \mathcal{C}_α denote the class of all sets $M \in \mathcal{C}$ such that

$$M = \bigcap_{n=1}^{\infty} (r_n \cup \mathcal{G}_n),$$

for some real r_n and some function f_n of Baire class α on \mathcal{S} . Let \mathcal{C}_α denote the class of all sets $N \in \mathcal{C}$ such that

$$N = \bigcap_{n=1}^{\infty} (r_n \cup \mathcal{G}_n),$$

for some real r_n and some function f_n of Baire class α on \mathcal{S} . It is easily shown that $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_\alpha \subset \mathcal{C}_{\alpha+1}$.

Let

$$\mathcal{C}_\alpha - \mathcal{C}_\beta = \mathcal{C}_\alpha \setminus \mathcal{C}_\beta,$$

$$\mathcal{G}_\alpha = \mathcal{G}_\alpha,$$

$$\mathcal{C}_\alpha^a = \mathcal{C}_\alpha,$$

$\mathcal{C}_\alpha^* = \mathcal{C}_\alpha$, If \mathcal{C} is any class of sets, let \mathcal{C}_α denote the class of all countable unions of members of \mathcal{C} , and let \mathcal{C}_α denote the class of all countable intersections of members of \mathcal{C} . Each of the following facts is either explicitly stated in¹², or can be easily deduced from statements found in¹³, or is obtained by a routine transfinite induction argument.

$$\text{I. } \mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \vee \beta}, \mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \wedge \beta} \bullet$$

$$\mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \wedge \beta}$$

II. Let A be any subset of the metric space \mathcal{S} . If f is a function of Baire class α on \mathcal{S} , then $f|_A$ is a function of Baire class α on A .

III. Let f be a function of Baire class α whose domain contains $\{(x, b) \mid x \in \mathcal{S}\}$. Then $f((x, b))$ is a function (of x) of Baire class α .

IV. If $A \in \mathcal{C}$, then

$$\mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \vee \beta}, \mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \wedge \beta}.$$

V. If f is of Baire class α on \mathcal{S} , then for each real r ,

and

$$\text{VI. If } \alpha \geq 2, \text{ then } (\mathcal{G}_\alpha)_\alpha \cup \mathcal{C}_\alpha = \mathcal{C}_\alpha \setminus \mathcal{C}_\alpha.$$

$$\text{VII. } \mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \vee \beta} \text{ if } \mathcal{C}_\alpha \mathcal{C}_\beta = \mathcal{C}_{\alpha \wedge \beta}.$$

VIII. \mathcal{C}_α and \mathcal{C}_β are closed under finite unions and intersections. \mathcal{C}_α is closed under countable unions and \mathcal{C}_α is closed under countable intersections.

309.

¹² F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

¹³ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

IX. Let f be a real-valued function on S . Suppose that for every real r and

$r > 0, +\infty) \in \mathcal{E}$.

Then f is of Baire class α .

Definition. If A and B are two sets, we will call A and B *equivalent*, and write $A \sim B$, if and only if $A - B$ and $B - A$ are both countable. It is easily verified that \sim is an equivalence relation.

Lemma 5. If $A \sim E$, then $\delta - A \sim \delta - E$ for any set δ . If $A_n \sim E_n$ (for all n in some countable set V), then

$\bigcup A_n \sim \bigcup E_n$ and $\bigcap A_n \sim \bigcap E_n$.

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The proof of this lemma is routine.

Definition. An interval of real numbers will be called *nondegenerate* if it contains more than one point.

Lemma 6. Any union of nondegenerate intervals is equivalent to an open set.

Proof. Let \mathcal{I} be a family of nondegenerate intervals and let $H = \bigcup \mathcal{I}$. For any x and y let $ZO, y) = [x, y]$, if $x \leq y$,

and let

$i(\%>y) = [y, \infty)$,

Define a relation $(R$ on H by

$x(Sly \iff I(x, y) \subset H$,

$(x, y) \subset H$).

It is easy to show that $(R$ is an equivalence relation on H . In view of the fact that a set A of real numbers is an interval if and only if

$x, y \in A \implies I(x, y) \subset A$,

it is obvious that each equivalence class is an interval. For each $x \in H$, there exists an $I \in \mathcal{I}$ with $x \in I$. Every member of I is equivalent to x . Thus each equivalence class contains more than one point, and hence is a nondegenerate interval. Let $\{J_a\}$ be the family of equivalence classes. Any disjoint family of nondegenerate intervals is countable, so there are only countably many J_a 's. Let E be the set of all endpoints of the various J_a 's. Then E is countable and

$H = \bigcup J_a \sim \bigcup J_a - E = \bigcup J_a^*$, $a \in A$

where J_a^* is the interior of J_a . This proves the lemma.

Lemma 7. Let h be an increasing real-valued σ -function on a nonempty set $E \subset \mathbb{R}$. Suppose that $|x - h(x)| \leq 1$ for every $x \in E$. Then h can be extended to an increasing real-valued function h_x on \mathbb{R} .

Proof. Let $e = \inf E$ (e may be $-\infty$). For each $x_Q \in (e, +\infty)$ set

$h(x_Q) = \sup \{h(x) \mid x \in (-\infty, x_Q) \cap E\}$.

Since $|x - h(x)| \leq 1$ for all $x \in E$,

$x \in (-\infty, x_Q] \cap E \implies h(x) \leq x_Q + 1$, so h is finite-valued. If $e = -\infty$ we are done. If

$e > -\infty$ then $x \in E$ implies

$h(x) \geq e - 1$, so A is bounded below. For $x_Q \in (-\infty, e)$ set

$\text{Ai}(\wedge o) = \text{g.l.b. } \{A(x) \mid xtE\}$,

It is easily verified that h_r has the desired properties.

Lemma 8. Let f be a real-valued function of Baire class a on R . Let h be an increasing real-valued function on R . Set $g(x) =$ Then there exists

a countable set N such that g is of Baire class a .

Proof. It is well known that an increasing function has at most countably many discontinuities. Let M be the set of discontinuity points of h . If f is of Baire class 0, then g is continuous at all points of $I \setminus M$, so g is of Baire class 0. This proves the lemma for the case $a = 0$.

We now proceed by transfinite induction. Suppose the lemma holds for every ordinal $X < a$. If f is of Baire class a we may choose a sequence of functions $\{f_n\}$ converging to f , where f_n is of Baire class $X_n < a$. If we set $g_n(x) = f_n(h^{-1}(x))$ it is clear that $g_n(x) \rightarrow f(x)$. By the induction hypothesis we may

choose (for each n) a countable set N_n such that g_n is of Baire class X_n . Let $N = \bigcup N_n$. Then $g_n \upharpoonright_{R \setminus N}$ is of Baire class X_n , and since $g_n \upharpoonright_{N^c} \rightarrow g \upharpoonright_{N^c}$, $g \upharpoonright_{N^c}$ is of Baire class a . This proves the lemma.

Theorem 4. Let f be a real-valued function of Baire class $a + 1$ on D° and let $\langle p \rangle$ be a finite-valued boundary function for f . Then $\langle p \rangle$ is of Baire class $a + 1$.

Proof. Let r and t be two real numbers with $r < t$. r and t will remain fixed throughout the first part of the proof. Set

$$Q = \{x \in D^\circ \mid \langle p \rangle(x) = r\},$$

$$E = P \setminus Q,$$

$$t - r \in \mathbb{R}.$$

Observe that $P \setminus Q = \{x \in D^\circ \mid \langle p \rangle(x) > r\}$. For each xE , choose an arc y_x at x such that $\lim_{z \rightarrow x} f(z) = \langle p \rangle(x)$, $y_x \cap \{z \mid |z - x| < \delta\} \neq \emptyset$, $z \rightarrow x$

and

$$(a) \quad \langle p \rangle(x) \in (r - \epsilon, r + \epsilon), \text{ if } x \in P$$

$$(b) \quad \langle p \rangle(x) \in (J - \epsilon, J + \epsilon), \text{ if } x \in Q.$$

(This is accomplished by cutting the arc off sufficiently close to a .) We remark that if $x \in P$ and $y \in Q$, then $y_x \cap y_y = \emptyset$.

We will say that y_x meets y_y in A_n^Q provided that y_x and y_y have subarcs y'_x and y'_y respectively such that $x \in y'_x \cap y'_y \cap A_n^Q$, $y'_x \cap y'_y \cap A_n^Q \neq \emptyset$, and $y'_x \cap y'_y \cap A_n^Q \neq \emptyset$. Let

$$L_0 = \{x \in P \mid (\forall n)(\exists y)(y \in A_n^Q)(y_x \text{ meets } y_y \text{ in } A_n^Q)\},$$

$$L_r = \{x \in Q \mid (\forall n)(\exists y)(y \in A_n^Q)(y_x \text{ meets } y_y \text{ in } A_n^Q)\},$$

$$M_0 = \{x \in P \mid (\exists n)(y_x \text{ meets no } y_y (y \in A_n^Q) \text{ in } A_n^Q)\},$$

$$\{x \in Q \mid (\exists n)(y_x \text{ meets no } y_y (y \in A_n^Q) \text{ in } A_n^Q)\},$$

$$L = L_0 \cup L_r,$$

$$M = M_0 \cup M_r.$$

Observe that L_0, L_r, M_0, M_r are pairwise disjoint, and that $P = L_0 \cup M_0$ and $Q = L_r \cup M_r$.

For each $x \in M$, let n_x be an integer such that y_x meets no y_y (with $y \in A_{n_x}^Q$) in $A_{n_x}^Q$. Notice that $n_x \rightarrow \infty$ implies y_x meets no y_y in A_{∞}^Q . Let

$K_n = \{x \in E \mid y_x \text{ meets } C(\mathbb{R}), \text{ and if } x \in M, n_x = n\}$.

Clearly $E = \bigcup_{n=1}^{\infty} K_n$. Moreover, $K_n \subset K_{n+1}$ for each n .

Take any fixed integer n . For each $x \in L_0$ we can find a $y \in E$ such that y_x meets y_y in $A(\mathbb{R})$. Let I_x be the nondegenerate closed interval between x and y . We shall show that $I_x \subset L_0 \cup (C(\mathbb{R}) - K_n)$. If $t \in I_x$, either $t \in C^Q - K_n$ or $t \in K_n$. Suppose $t \in K_n$. Then y_t meets $C(\mathbb{R})$, and (if $t \in M$) $n_t = n$. It is clear from Figure 1 that y_t must meet either y_x or y_y in $A(\mathbb{R})$. (This can be rigorized by means of Theorem 11.8 on p. 119 in¹⁴.)

Consequently, $t \notin M$. Now $x \in L_0 \subset Q \subset P$, so since y_x intersects y_y , $y \in E - Q = P$. Similarly, since y_t meets y_x or y_y , $t \in E - Q = P$. Thus $t \in P - M = L_0$. We have shown that $t \in I_x$ implies that $t \in C^Q - K_n$ or $t \in L_0$, so $I_x \subset L_0 \cup (C^Q - K_n)$. It follows that (for each n)

$$L_0 \cup (C^Q - K_n) \subset E \cap \{x \mid x \in C^Q - K_n\} \cap E.$$

Let $W_n = L_0 \cup (C^Q - K_n)$. By Lemma 6, W_n is equivalent to an open set. $l_0 \cup (C^Q - K_n) \cap E$

$$\begin{aligned} C \cap [L_0 \cup (C^Q - K_n)] &\cap E = \{l_0 \cup (C^Q - K_n)\} \cap E \\ &= \{L_0 \cap E\} \cup \{(C^Q - K_n) \cap E\} = L_0 \cup (C^Q - K_n). \end{aligned}$$

Therefore $L_0 = W_n \cap E$. Since each W_n is equivalent to an open set there exists a $G_Q \in \mathcal{G}_8$ such that

$$L_0 \subset G_Q \cap E.$$

Similar reasoning shows there exists a $G_x \in \mathcal{G}_8$ such that

$$E \subset G_x.$$

Next we study the properties of M_Q . It is convenient to define a function $\gamma : R^2 \rightarrow R$ by $\gamma(x, y) = \text{If } x \in M \cap C \setminus K_n$, then, starting at x and proceeding along y_x , let $\gamma_n(x)$ be the first point of C^Q_n reached. Set $\gamma(x) = \gamma_n(x)$ (for $x \in M \cap C \setminus K_n$).

γ is an increasing function on $M \cap C \setminus K_n$; for if $x_1, x_2 \in M \cap C \setminus K_n$ and $x_1 < x_2$, then, since cannot meet y_{x_1} in A^Q_n , it is evident (see Figure 2) that $\gamma(x_1) < \gamma(x_2)$. (The argument can be rigorized by means of Theorem 11.8 on p. 119 in¹⁵.) Since

$$y_x \in \{z \mid |z - 1|, |x - \gamma_n(x)| \leq 1\}.$$

So by Lemma 7 can be extended to an increasing function h_n on C^Q .

Let

$$G_n(x) = \bullet$$

For $x \in M \cap C \setminus K_n$,

$$g_n(x) =$$

If $x \in M$, then for all sufficiently large n , $x \in M \cap C \setminus K_n$, so $\lim g_n(x) = \lim f(\gamma_n(x)) = \gamma(x)$. $\mathbf{n} \rightarrow \mathbf{a} \rightarrow \mathbf{n} \rightarrow \mathbf{a}$

Thus $g_n \rightarrow \gamma$. By III, γ is a function (of x) of Baire class a , so by Lemma 8 we can choose, for each n , a countable set N_n such that $g_n \upharpoonright_{C^Q - N_n}$ is of Baire

¹⁴ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

¹⁵ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

class a . Let $N = \bigvee_{j \in \mathbb{N}} N_j$. Then g_n is of Baire class a . But $g_n \rightarrow g$ pointwise, so g is of Baire class $a + 1$.

Now

$P(M \cap N) = \{x \in X : (x \in M) \rightarrow (x \in N)\}$, where $T \in \mathcal{B}^{a+1}$ (by IV and V). Clearly $P(C \setminus M) \sim T(C \setminus M)$.

We have

$$L = P_0 \cup L, \quad (G_0 \cap F) \cap J(G_1 \cap E) = (G_0 \cup G_1) \cap H \cap E,$$

so $L \sim G \cap C \setminus E$ where $G \in \mathcal{G}_8$. Also

$$M_0 = PKM \cap TC \setminus M = T(C \setminus (E \cap L))$$

$$\sim TC \setminus [E \cap (G \cap P)] = [T \cap (C^\circ \cap G)] \cap E.$$

Since $G \in \mathcal{G}_5$, $G^\circ \cap G \in \mathcal{G}_5$, so by VI and VIII, $T(C \setminus (C^\circ \cap G)) \in \mathcal{B}^{a+1}$. Thus

$$M_0 \sim T_0 \cap E,$$

where $T_0 \in \mathcal{B}^{a+1}$. Now we can examine the properties of P .

$$P = (G_0 \cap KE) \cup (T_0 \cap E) = (G_0 \cup T_0) \cap E,$$

so, again by VI and VIII,

$$P \sim \mathcal{B}^{a+1} \cap P,$$

where $T_x \in \mathcal{B}^{a+1}$. Since a countable set is in and the complement of a countable set is in \mathcal{G}_8 , it is easy to show (using VI and VIII) that

$$P = T_2 \cap P,$$

where $T_2 \in \mathcal{B}^{a+1}$. Since $P \cap Q = \{f\}$

$$P \not\subseteq T_2 \cap C^\circ \cap Q.$$

Remembering the definitions of P and Q and observing the fact that $C^\circ \sim \bigcap_{n \in \mathbb{N}} (K_n \cap \mathcal{B}^{a+1})$, we can summarize the results of the first part of the proof as follows.

For each pair r, t of real numbers with $r < t$, there exists a set $T(r, t) \in \mathcal{B}^{a+1}$ such that

$$\bigcap_{n \in \mathbb{N}} (T(r, t) \cap C_n) = \emptyset.$$

Given any real r , let $\{Z_n\}$ be a strictly decreasing sequence of real numbers converging to r . Then

$$\bigcap_{n \in \mathbb{N}} (Z_n \cap \mathcal{B}^{a+1}) = \emptyset.$$

So

$$\bigcap_{n \in \mathbb{N}} (Z_n \cap T(r, Z_n)) \subseteq \bigcap_{n \in \mathbb{N}} (Z_n \cap T(r, Z_n)) = \emptyset,$$

and hence

$$\bigcap_{n \in \mathbb{N}} (Z_n \cap T(r, Z_n)) = C \setminus T(r, Q).$$

By VIII,

$$\bigcap_{n \in \mathbb{N}} (Z_n \cap T(r, Z_n)) \in \mathcal{B}^{a+1}.$$

Since f is an arbitrary function of Baire class a in Z and g is an arbitrary boundary function for f , we can replace f, g, r by h, h, cr, r to find that

$$\bigcap_{n \in \mathbb{N}} (Z_n \cap \mathcal{B}^{a+1}) \in \mathcal{B}^{a+1}.$$

Also,

$$\bigcap_{n \in \mathbb{N}} (Z_n \cap \mathcal{B}^{a+1}) = C^\circ \cap \mathcal{B}^{a+1}.$$

By IX, g is of Baire class $a + 1$. *Q.E.D.*

5. Boundary functions for measurable functions.

Theorem 5. Let f be a real-valued Borel-measurable function in D^Q and let φ be a finite-valued boundary function for f . Then φ is Borel-measurable.

Since every Borel-measurable function is of some Baire class α , this theorem is an immediate consequence of Theorem 4. We now show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable.

Let μ denote Lebesgue measure on R and let μ^2 denote Lebesgue measure on R^2 . Let μ^* denote exterior Lebesgue measure on R that is,

$$\mu^*(E) = \text{g.l.b. } \{ \mu(G) \mid G \text{ is open and } E \subset G \},$$

for any set $E \subset R$.

Lemma 9. Let h be an increasing real-valued function on a set $E \subset R$. Then there exists an open interval $I \supset E$ such that h can be extended to an increasing real-valued function on I .

Proof. If E is unbounded below, set $a = -\infty$. If E is bounded below, set $a = \text{g.l.b. } E$, if $(\text{g.l.b. } E) \notin E$,

$$a = (\text{g.l.b. } E) - 1, \text{ if } (\text{g.l.b. } E) \in E.$$

If E is unbounded above, set $b = +\infty$. If E is bounded above, set

$$b = \text{l.u.b. } E, \text{ if } (\text{l.u.b. } E) \notin E,$$

$$b = (\text{l.u.b. } E) + 1, \text{ if } (\text{l.u.b. } E) \in E.$$

Let $I = (a, b)$. Clearly $E \subset I$. Let $e = \text{g.l.b. } E$ (e may be $-\infty$). For $x_0 \notin (e, b)$ set

$$f(x) = \text{l.u.b. } \{ h(x) \mid x \in (a, x_0] \cap E \}.$$

If $e = a$ we are done. If $e > a$ then $e \notin E$. For $x_0 \notin (a, e]$ set $f(x_0) = A(e)$. It is easily verified that f is finite-valued and increasing, and is an extension of h .

Lemma 10. Let $E \subset R$ be a set of measure 0 and let h be an increasing function on E . Suppose $h(E)$ has measure 0. Then $\{x + h(x) \mid x \in E\}$ has measure 0.

Proof. Extend h to an increasing function g on an open interval $I = (a, b) \supset E$. Set $g(a) = -\infty$ and $g(b) = +\infty$. Take any $\epsilon > 0$. Choose an open set G such that I and $\mu^*(G) < \epsilon/2$. Choose an open set $H \supset h(E)$ with $\mu^*(H) < \epsilon/2$.

Say

$$G = \bigcup I_n, \text{ and } H = \bigcup J_m, \text{ where } \{I_n \mid n \in \mathbb{N}\} \text{ and } \{J_m \mid m \in \mathbb{N}\} \text{ are countable families of disjoint open intervals.}$$

Let $I_n = (a_n, b_n)$, and observe that $a_n, b_n \in [a, b]$. Set

$$S = \{ (n, \{g(a_n), g(b_n)\}) \mid n \in \mathbb{N} \}.$$

\mathbb{N}

Notice that S is countable. Set

$$K_n = (g(a_n), g(b_n)).$$

One can easily verify that $k \in \mathbb{N}$ implies $K_k \cap K_n = \emptyset$.

If A and B are two subsets of \mathbb{R} , let

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

It is easy to show that for any two intervals J and $\langle 7', g_e(\langle 7 + \langle /' \rangle) g + \wedge(J') \rangle$. Let $W = \{x + h(x) \mid x \in E\}$.

Assertion.

$w \in (E + S) \cap J \setminus J \cap [(Z_n \cap g(W) \setminus (J_n \cap K_n))]$. $mN \cap mM$

To prove this, let w be an arbitrary point of W . Write $w = x + h(x)_f$ where $x \in E$. For some n , $x \in I_n$. Since g is increasing,

$h(x) = g(x) \in [\# \langle \rangle, g(b_n)]$.

If $h(x)$ equals $g(a_n)$ or $\#(6_n)$, then $h(x) \in S$, so $w = x + h(x) \in E + S$. On the other hand, suppose $h(x) \neq \#(a_n)$, $\langle ?(\&, \cdot) \rangle$. Then $h(x) \in K_n$. Also, $g(x) = h(x) \in J_m$ for some m . Thus $h(x) \in J_m \cap K_n$ and $x \in I_n \cap C \setminus g^{-1}(J_m)$ so that

$w = x + h(x) \in (I_n \cap C \setminus g^{-1}(J_m)) \cap (J_m \cap K_n)$.

This proves the Assertion.

Since g is increasing, $g \wedge J \wedge$ is an interval, so both $I_n \cap C \setminus g^{-1}(J_m)$ and $J_m \cap K_n$ are intervals. Also note that $m \neq I$ implies $g^{-1}(J_m) \cap g^{-1}(J_i) = \emptyset$. By the Assertion,

$M \cap (g + s) \cap S \cap g^{-1}(J_m) \cap (J_n \cap K_n) = \emptyset$ ***mN mtM***

$g \cap n_e(E + s) \cap E \cap [m \setminus b \cap n \cap o$

mN mzM

$= m \setminus U(s + t)) \cap E \cap [E \cap g^{-1}(J_m) \cap \mathbb{R} \setminus (J_m \cap K_n)]$ ***szS mN mtM mtM***

$\langle E \cap \langle em \rangle + E \rangle \cap \langle em \rangle \cap E \cap W \cap E \cap mGA \cap n \cap 2Q]$

utS mN mtM

$= 0 \cap ju(G) \cap E \cap C \setminus K_n$ ***nzN mzM***

$= m(g) \cap E \cap m(A \cap K_n)$

$mtM \cap nzN$

$m(G) \cap E \cap mW = m(\langle ? \rangle) \cap \mathbf{1EH} \cap \langle e \rangle$ ***mt M***

Since e is arbitrary, $jn_e(W) = 0$.

Lemma 11. Let $L = \{(x, a) \mid x \in E\}$ and $M = \{(x, b) \mid x \in R\}$ be two horizontal lines in \mathbb{R}^2 . Let E be a set of (linear) measure 0 in L and let F be a set of (linear) measure 0 in M . Let \mathcal{L} be a set of closed line segments such that

(a) $\mathcal{L} \subset \mathbb{R}^2$, $S \cap \mathcal{L} = \emptyset$ if $S \cap F = \emptyset$

(b) $\mathcal{L} \subset \mathbb{R}^2 \Rightarrow$ one endpoint of s lies in E and the other endpoint lies in F .

Let $S = \bigcup_{k \in \mathbb{N}} s_k$. Then $u^2(S) = 0$.

Proof. Assume without loss of generality that $b > a$. For any $(x, y) \in \mathbb{R}^2$ let $\wedge(\{x, y\}) = x$. For any $y \in R$ let $l_y = \{(x, y) \mid x \in R\}$. Let

$E_o = \{z \in E \mid z \text{ is the endpoint of some } s \in \mathcal{L}\}$,

and observe that E_o has linear measure 0. For any set $A \subset \mathbb{R}^2$ we of course set $tt(A) = \{x \in R \mid (x, y) \in A \text{ for some } y \in R\}$.

We define a function h on $ir(E_o)$ as follows. If $s \in \mathcal{L} \cap (Eq)$, then $\{x, a\} \subset E_o$, so we can choose a (unique) segment $s \subset \mathbb{R}^2$ with one endpoint at (x, a) . If the other endpoint of s is p , we set $h(x) = 7r(p)$. Clearly h maps $ir(E_o)$ into $tt(F)$.

Since the segments in \mathcal{L} cannot intersect each other, h must be an increasing function.

Take any y_Q with $b > y_Q > a$. Let $c = b - y_0$, $d = y_0 - a$. A simple computation shows that if $q \in I_{V_0} \setminus C \setminus \delta$, then

$$\begin{aligned} \text{tt}(q) = \\ \text{ex} + dh(x) \cdot c + d \end{aligned}$$

for some $x \in \text{tt}(\mathcal{L}_{?_0})$. So

$$\begin{aligned} 7r(I_{V_0} \setminus A \setminus \delta) \subset C \\ \text{ex} + dh(x) \cdot e + d \\ X \not\subset 7r(\text{Fo}) \end{aligned}$$

Now $(d/e)h(x)$ is an increasing function mapping $\text{tt}(E_q)$ into $(d/c)7r(F)$, so by Lemma 10

$$\begin{aligned} x + h(x) \\ X \not\subset Tt(Eq) \end{aligned}$$

has measure 0. Hence

$$\begin{aligned} X \not\subset Tt(Eq) \\ \text{ex} + dh(x) \cdot e + d \\ X \not\subset Tt(E_0) \end{aligned}$$

has measure 0, so $A \cap SY = \emptyset$. But $A \cap \delta = \emptyset$ also when $y_Q \notin (a, b)$, so $A \cap \delta = \emptyset$ for every y . If we knew that δ were measurable, the lemma would follow immediately from the Fubini theorems. But since we have, as yet, no guaranty of the measurability of δ , a more complicated argument is necessary. At several stages in the argument the reader will find it useful to draw diagrams to help him visualize the situation.

For any $y_1, y_2 \in R$, let

$$U(y_1, y_2) = \{(x, y) \mid x, y \in R, y_1 < y < y_2\}.$$

A set of the form $U(y_1, y_2)$ will be referred to as a *horizontal open strip*.

For each positive integer n , let $\mathcal{L}(n)$ denote the set of all segments $s \in \mathcal{L}$ such that s has a point in common with $\{(x, y) \mid x \in (-n, n), y \in \delta\}$. Let

$$S(n) = \bigcup_{s \in \mathcal{L}(n)} s.$$

Since I_a and I_b have (plane) measure 0, and since

$$\text{sc } I_a \cup I_b \subset S(n), \quad n = 1, 2, \dots$$

it is sufficient to show that each $S(n)$ has measure 0.

Let n be a fixed positive integer. Set $a^* = a + 1/n$ and $b^* = b - 1/n$. Take any $\epsilon > 0$. Choose ϵ_0 so that $2\epsilon_0 + \epsilon^* < \epsilon/(b - a)$. Let y_0 be any member of $[a^*, b^*]$. For the time being, y_Q will be held fixed.

For each $s \in \mathcal{L}$, let p_s be the endpoint of s on I_b , let q_s be the intersection point of s with I_{V_0} , and let r_s be the endpoint of s on I_a .

Choose an open set $G \subset R$ such that $\text{tt}(I_{V_0} \cap S(n)) \subset G$ and $\text{g}((?) < \epsilon_0$. Say $G = \bigcup_{j=1}^m J_j$, where $J_j = (a_j, b_j)$ and the J_j 's are pairwise disjoint. We may assume that each J_j contains a point of $\text{tt}(I_{V_0} \cap S(n))$. For each let

$Ci = g.l.b. \{7r(p_a) \mid s \in \mathcal{L}(n), 7r(g_a) \in \mathbb{Z}J, \\$
 $dj = l.u.b. \{\wedge(p,) \mid s \in \mathcal{L}(n), 7r(g_{\ll}) \in \mathbb{Z},\}, c< = g.l.b. \{?r(r_a) \mid 8 \in <\mathcal{L}(n), ?r(\wedge) \in \mathbb{Z},\}, d'i = l.u.b. \{?r(r_a) \mid 8 \in \mathcal{L}(n), ir(q_s) \in \mathbb{Z}J.$

Note that $c, g d,-$ and $c< g d< .$ Since the segments in \mathcal{L} cannot intersect each other, it is easily seen that the intervals $(c,-, d,)$ are all pairwise disjoint. It is also clear (from the definition of $\mathcal{L}(n)$) that each $(c,-, d,)$ is a subset of $(\sim n, ri)$. Hence, if we set $a,- = d,< c_{\gamma-}$, we have $22,- a,- g 2n$.

For each j , let $s(j)$ be the line segment joining the two points $(c<, a), (, b)$, and let $t(j)$ be the line segment joining the two points $(d<, a), (d,-, b)$. Let A_f be the closed subset of $U(a, b)$ which is enclosed by the two line segments $80), \wedge 0)$. Let Hj denote the intersection of $A,-$ with the horizontal open strip

$$V = \text{LI max} \\ \epsilon_0 I \bullet J \gamma I \epsilon_0 I I \circ' \wedge - \wedge p^{\min} V' \wedge + 2 \wedge / r$$

Note that H_f is measurable. Setting $ZZ = H_f$, it is clear from the definition of the A/s that

$$S(ri) C \setminus V QH.$$

Take any $y \in R$. We wish to show that

$$m0-(ZZ A l_y)) < \\ _ _ e _ _ b - a$$

We can, of course, assume that

$$l J \epsilon_0 I \bullet J 7 . \epsilon_0 I \mid$$

$$y \in I \max \mid a, y_a - , \text{mm } S b, y_0 + \wedge(r$$

An elementary computation, using the geometrical properties of Hj , shows that

$$n i,)) g (i + 4 \wedge - \\ \mid o y_Q / o y_Q$$

Therefore

$$n z,,)) g E n w \\ - G + ^n \wedge)^{e_0} + 2n^2$$

$$'' 2c_0 + \epsilon_0 < t , b - a$$

so $v(ir(H l_v Y) < e/(b - a)$ for every y .

We have shown that for each $y_Q \in [a^* \&^*]$ there exists a horizontal open strip

$V(y_Q)$ containing l_{V_0} , and there exists a measurable set $H(y_Q) \subset y(?/_0)$, such that

$$S(n) H V(y_0) \mathcal{L} H(y_0)$$

and (for every y) $ir(H(y_Q) l_v)$ is measurable and

$$y \ll H(y_0) \subset y Z_y)) <$$

The various open strips $VG/o) (i/_0 \ll [a^*, 6^*])$ clearly cover the compact set $\{(0, y\} \cup y^e \mathbb{P}^> \mathcal{E}\}$ • Choose a finite subcovering $V(y_2 \mid \bullet \bullet \bullet , V(y_m)$. Set

$$tm - 1 /$$

$$m \setminus$$

$$\cup V(y,)]$$

$$J^{\ll} + 1 /$$

$$n I 7(a^*, 6^*).$$

$H(y_m) \cup \{J \text{ IW}\} -$
 X

Obviously K is measurable, and for each y , $\text{ir}(K \text{ A } Z,)$ is measurable and $C \setminus l_v)) < e/(b - a)$. Moreover, $S(n) \subset K$. We have

$$\lambda(K) = \int (K \setminus l_v) dy \leq \int (b - a^*) dy = (b - a^*) < e.$$

Since e is arbitrary, this shows that

$$\text{g.l.b. } \{\lambda(X) \mid K \text{ measurable, } S(n) \subset K\} = 0.$$

Therefore $S(n)$ has measure 0.

Lemma 12. For every $e > 0$ there exists a strictly increasing function h on R such that $h(R)$ has measure 0, and for every x , $|h(x) - h(y)| < e$

Proof. For each (not necessarily positive) integer n , let $I_n = [ne, (n+1)e]$. Then $I_n \cap R = \emptyset$. There exists a strictly increasing function $f : [0, 1] \rightarrow [0, 1]$

such that $m(f([0, 1])) = 0$. For example, such a function may be defined as follows.

Any number in $[0, 1)$ may be written in the form

$$.a_1a_2a_3 \dots a_n \dots, \text{ (binary decimal),}$$

where the decimal does not end in an infinite unbroken string of 1's. Set

$f(.a_1a_2a_3 \dots a_n \dots) = .b_1b_2b_3 \dots b_n \dots$, (ternary decimal), where $b_i = 0$ if $a_i = 0$ and $b_i = 2$ if $a_i = 1$. Set $f(1) = 1$. f maps $[0, 1]$ into the Cantor set, so $m(f([0, 1])) = 0$. It is easily shown that f is strictly increasing.

For each n , it is easy to obtain from f a function $f_n : I_n \rightarrow I_n$ such that f_n is strictly increasing and $m(f_n(I_n)) = 0$. Set

$$h(x) = f_n(x) \text{ for } x \in (ne, (n+1)e].$$

There is no difficulty in proving that h has the desired properties.

Theorem 6. Let be an arbitrary function on $C^Q = \{f(x, 0) \mid x \in R\}$. Then there exists a function j on $D^0 = \{(x, y) \mid y > 0\}$ such that $j(z) = 0$ almost everywhere and j is a boundary function for f .

Proof. For each positive integer n let h_n be a strictly increasing function on R such that $y(h_n(R)) = 0$, and for every x , $|h_n(x) - h_n(x')| < 1/n$. Let E_n have linear measure 0. For each n , x let $s_n(x)$ be the line segment joining $(h_n(x), 1/n)$ and $(A_{n+1}(a), 1/(n+1))$. Since

E_n is a subset of

$$h_n(x) > h_n(x') \implies x > x' \implies h_{n+1}(x) > h_{n+1}(x')$$

we find that $x = x'$ implies $s_H(x) \subset s_H(x')$. Since each $s_n(x)$ has one endpoint in E_n and the other in E_{n+1} , Lemma 11 shows that for each n

$$m(s_n(x)) = 0.$$

Hence

Hence

$$m^2(\bigcup_{n=1}^{\infty} s_n(a)) = 0. \text{ 'n=sl } \mathbf{xtR} \bullet$$

Set

$$j(z) = \begin{cases} f(z, 0), & \text{if } z \in s_n(x) \text{ for some } n, \\ 0, & \text{if } z \text{ is not in any } s_n(x). \end{cases}$$

$$j(z) = 0, \text{ if } z \text{ is not in any } s_n(x).$$

$$j(z) = 0 \text{ almost everywhere. Let}$$

$$y(x) = \{\{x, 0\}\} \cup 0s_n(x). \quad 74 = 1$$

Since the endpoints of $s_n(\$)$ are at $(A_n(z), 1/n)$ and $(h_{n+1}(x)j \ l/(n+1))$, and since $(h_n(x), 1/n) \rightarrow (x, 0)$ as $n \rightarrow \infty$, it is clear that $y(x)$ is an arc at $(x, 0)$. Obviously $\lim j(z) = \langle p(\{x_t, 0\})$.

This proves the theorem.

Corollary. *There exists a measurable function in D° having a nonmeasurable boundary function.*

6. Concluding remarks. Our theorem on boundary functions for continuous functions could have been proved by a small modification of the argument in Section 4, but the proof in Section 3 is shorter and neater.

The reader may wonder whether Theorem 4 holds true for functions taking values on the Riemann sphere as well as for real-valued functions. The theorem does, in fact, remain true in the sphere-valued case. If we regard the Riemann sphere 2 as a subset of R^3 and apply Theorem 4 to each component of f and $\langle p$, we find that $\langle p$ is of Baire class $a+1$ with R^3 as the universal range space. It is then easy to show by means of Satz 2 in Banach's paper¹⁶ that $\langle p$ is of Baire class $a+1$ with 2 regarded as the universal range space. A similar procedure shows that Theorem 5 also remains true for functions taking values on the Riemann sphere.

The results of Sections 2, 3 and 4 cannot be extended to three dimensions—at least not in the most obvious way. We can show this as follows. Let K be an open cube in R^3 and let F be one face of K . If f is defined in A , then we say $\langle p$ (defined on F) is a *boundary function* for f provided that for each $x \in F$ there exists an arc y with one endpoint at x such that $y \subset K$ and

$$\lim f(v) = (p(x), v \rightarrow \widehat{R}) \ vzy$$

Lemma 13. *Suppose that every point of F is an ambiguous point of the function $f : K \rightarrow R^3$. Then f has a nonmeasurable boundary function.*

Proof. Let E be a nonmeasurable subset of F . Since each point of F is an ambiguous point we can choose, for each $x \in F$, two distinct points $\hat{\gamma}(x), \langle p_2(x) \in R^3$ such that there exist arcs y_i at x with

$$\lim f(p) = P_i(x), \quad (i = 1, 2). \quad v \rightarrow x \ vsy \ i$$

Let

$$\langle p(x) = \langle p_1(x), \text{ if } x \in E,$$

$$\langle p(x) = \langle p_2(x), \text{ if } x \in F - E.$$

Then

$$\langle p(x) - \hat{\gamma}(x) = 0, \text{ if } x \in E,$$

$$\langle p(x) - \hat{\gamma}(x) \neq 0, \text{ if } x \in F - E.$$

Therefore $(\langle p - \hat{\gamma})^{-1}(\{0\}) = E$, so $\langle p$ is not a measurable function. Hence either $\langle p$ or $\langle p_x$ is a nonmeasurable function. Since $\langle p$ and $\langle p_Y$ are both boundary functions for f , the lemma is proved.

¹⁶ S. Banach, Über analytisch darstellbare Operationen in abstrakten Räumen, *Fund. Math.*, 17 (1931) 283-295.

P. T. Church¹⁷ has constructed an example of a homeomorphism f from K onto K such that every point of F is an ambiguous point for f . By Lemma 13, f has a nonmeasurable boundary function $\langle p$. Theorem 1 is therefore false in three dimensions. Write f and $\langle p$ in terms of their components; say $f = (f_1, f_2, f_3)$ and $\langle p = \langle p_1, p_2, p_3 \rangle$. Since $\langle p$ is nonmeasurable, one of its components, say p_1 , is nonmeasurable. But p_1 is a boundary function for the continuous real-valued function f_1 , so Theorem 2 and Theorem 4 must be false in three dimensions.

References

The University of Michigan

¹⁷ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

5. 1966 - On a Boundary Property of Continuous Functions

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Explanation by John D. Bullough

The author generalizes the result of McMillan (1966) to the effect that the set of curvilinear convergence of a continuous function f from D into Z is of type $F(sd)$. The generalization considers f as a continuous function from D into a compact metric space E . Topologizing the set of closed sets $C(E)$ of E with the Hausdorff metric and letting E be any closed set in $C(E)$, it is shown that the set of all $x \in C$ such that there is a boundary path v at x with the cluster set of f along v contained in some set of E is of type $F(sd)$. Taking E to be the set of all singletons $\{y\}$, $y \in E$ (which is closed in $C(Z)$) McMillan's theorem is obtained.

Various other corollaries are given by selecting appropriate closed sets $E \in C(E)$.

Article by Ted

On a Boundary Property of Continuous Functions

T. J. Kaczynski

Let D be the open unit disk in the plane, and let C be its boundary, the unit circle. If x is a point of C , then an *arc at* x is a simple arc y with one endpoint at x such that $y - \{x\} \subset D$. If f is a function defined in D and taking values in a metric space K , then the *set of curvilinear convergence* of f is

$\{x \in C \mid \text{there exists an arc } y \text{ at } x \text{ and there exists a point } p \in K \text{ such that } \lim_{z \rightarrow x} f(z) = p\}.$

$Z \rightarrow X$ zey

J. E. McMillan proved that if f is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of f is of type F_a [2, Theorem 5]. In

this paper we shall provide a simpler proof of this theorem than McMillan's, and we shall give a generalization and point out some of its corollaries.

Notation. If S is a subset of a topological space, \bar{S} denotes the closure and S^* denotes the interior of S . Of course, when we speak of the interior of a subset of the unit circle, we mean the interior relative to the circle, not relative to the whole plane. Let K be a metric space with metric p . If $x_0 \in K$ and $r > 0$, then

$$S(r, x_0) = \{x \in K \mid p(x, x_0) < r\}.$$

An arc of C will be called *nondegenerate* if and only if it contains more than one point.

LEMMA 1. *Let be a family of nondegenerate closed arcs of C . Then $\bigcup_{I \in I^*} I$ is countable.*

Proof. Since $\bigcup_{I \in I^*} I$ is open, we can write $I^* = \bigcup_n J_n$, where $\{J_n\}$ is a countable family of disjoint open arcs of C . If

$$x_0 \in \bigcup_{I \in I^*} I, \text{ then } x_0 \in J_n \text{ for some } n.$$

then for some $I_0 \in I^*$, x_0 is an endpoint of I_0 . For some n , $I_0 \subset J_n$, so that $x_0 \in J_n$. But $x_0 \in J_n$, so that x_0 is an endpoint of J_n . Thus $\bigcup_{I \in I^*} I$ is contained in the set of all endpoints of the various J_n ; this proves the lemma. \square

In what follows we shall repeatedly use Theorem 11.8 on page 119 in¹ without making explicit reference to it. By a cross-cut we shall always mean a cross-cut of D . Suppose y is a cross-cut that does not pass through the point 0. If V is the component of $D - y$ that does not contain 0, let $L(y) = \bar{V} \cap C$. Then $L(y)$ is a nondegenerate closed arc of C .

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Suppose n is a domain contained in $D - \{0\}$. Let r denote the family of all cross-cuts y with $y \cap n \neq \emptyset$. Let

$$I(n) = \bigcup_{y \in r} L(y), \quad I_0(n) = \bigcup_{y \in r} L(y)^*.$$

$$y \in r, y' \in r$$

Let $\text{acc}(n)$ denote the set of all points on C that are accessible by arcs in n .

The following lemma is weaker than it could be, but there is no point in proving more than we need.

LEMMA 2. *The set $\text{acc}(n) - I_0(n)$ is countable.*

Proof. By Lemma 1, $I(n) - I_0(n)$ is countable; therefore it will suffice to show that $\text{acc}(n) - I(n)$ is countable. If $\text{acc}(n)$ has fewer than two points, we are done. Suppose, on the other hand, that $\text{acc}(n)$ has two or more points. If $a \in \text{acc}(n)$, then there exists $a' \in \text{acc}(n)$ with $a' \neq a$. Let y, y' be arcs at a, a' , respectively, with

$$y \cap n \neq \emptyset, y' \cap n \neq \emptyset.$$

Let p be the endpoint of y that lies in n , p' the endpoint of y' that lies in n . Let $y'' \subset n$ be an arc joining p to p' . The union of y, y' , and y'' is an arc α joining a to a' .

¹ S. Banach, Über analytisch darstellbare Operationen in abstrakten Räumen, *Fund. Math.*, 17

By², there exists a simple arc 0^1 co that joins a to a^\wedge . Clearly, 0^1 is a cross-cut with $0^1 \cap D \subset \mathbb{R}^n$ and $a, a^\wedge \in L(0^1)$. Thus $a \in I(n)$, and so $\text{acc}(n) \subset I(0)$. •

LEMMA 3. Suppose O_1 and O_z are domains contained in $D - \{O\}$. If

(1) $I_q(CO_j) \cap \text{acc}(O_1)$ and $I_q(Q_2) \cap \text{acc}(Q_2)$

are not disjoint, then n_1 and O_z are not disjoint.

Proof. We assume n_1 and O_z are disjoint, and we derive a contradiction. Let a be a point in both of the two sets (1). Let Y_i be a cross-cut with $Y_i \cap D \subset n_i$ such that $a \in L(y_i)^*$ ($i = 1, 2$). Let U_i and V_i be the components of $D - Y_i$, and (to be specific), let U_i be the component containing 0 . Note that $y_1 \cap D$ and $y_2 \cap D$ are disjoint.

Suppose $y_i \cap D \subset V_z$ and $yz \cap D \subset V_j$. Then, since $y_i \cap D \subset U_i$, U_i has a point in common with V_z . But $O \in U_1 \cap U_z$, so that U_1 has a point in common with U_z also. Since U_i is connected, this implies that U_i has a common point with $Y_z \cap D$, which contradicts the assumption that $Y_z \cap D \subset V_p$. Therefore $y_i \cap D \subset V_2$ or $Y_z \cap D \subset V_i$. • We conclude that either $y_1 \cap D \subset U_2$ or $y_2 \cap D \subset U_1$. By symmetry, we may assume that $Y_z \cap D \subset U_i$.

It is possible to choose a point $b \in L(y_i)^*$ that is accessible by an arc in O_z , because a is in the closure of $\text{acc}(O_z)$. Let y be a simple arc joining b to a point of $y_4 \cap D$, such that $y - \{b\} \subset O_z$. Then $y - \{b\}$ and y_i are disjoint. Also, $y - \{b\}$ contains a point of U_i (namely, the point where y meets $y_2 \cap D$); therefore $y - \{b\} \subset U_i$. Hence $b \in U_i$. Since $b \in L\{y_1\}^*$, this is a contradiction. •

THEOREM 1 (J. E. McMillan). Let K be a complete separable metric space, and let f be a continuous function mapping D into K . Let

$X = \{x \in \mathbb{R}^n \mid \text{there exists an arc } y \text{ at } x \text{ for which } \lim_{z \rightarrow x} f(z) \text{ exists}\} \cap \mathbb{R}^n$

Then X is of type F_a .

Proof. Let $\{p_k\}_{k=1}^\infty$ be a countable dense subset of K . Let $\{Q(n, \theta)\}$ be a counting of all sets of the form

where θ is a rational number. Let $\{U(n, m, k, \mathcal{L})\}_{n=1}^\infty$ be a counting (with repetitions allowed) of the components of

(We consider 0 to be a component of 0 .) Let

$A(n, m, k, \mathcal{L}) = \text{acc}[U(n, m, k, \mathcal{L})]$.

Set

$co \ co \ co \ co$

$Y = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \bigcup_{k=1}^\infty \bigcup_{f=1}^\infty I_0(U(n, m, k, \mathcal{L})) \cap A(n, m, k, \mathcal{L})$.

$n=1 \ m=1 \ k=1 \ f=1$

Since $I_0(U(n, m, k, \mathcal{L}))$ is open, it is of type F_a . It follows that Y is of type $F_{a < 5}$.

I claim that $Y \subset X$. Take any $y \in Y$. For each n , choose $m[n]$, $k[n]$, $f[n]$ with

(2) $y \in I_0(U(n, m[n], k[n], f[n])) \cap A(n, m[n], k[n], \mathcal{L}[n])$ ($n = 1, 2, 3, \dots$).

(1931) 283-295.

² P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

For convenience, set $U_n = U(n, m[n], k[n], f[n])$. By (2) and Lemma 3, U_n and U_{n+1} have some point z_n in common. For each n , we can choose an arc $y_n \subset U_{n+1}$ with one endpoint at z_n and the other at z_{n+1} . Then $y_n \subset Q(n+1, m[n+1])$. Also, and therefore

$$\sum_{n=1}^{\infty} P(p_k[n] - p_k[n+r]) \leq \sum_{n=1}^{\infty} P(p_k[n+i-1] - p_k[n+i]) < \infty^2$$

Thus $\{p_k[n]\}$ is a Cauchy sequence and must converge to some point $p \in K$. Because $r_n \subset U_{n+1} \subset f^{-1}(S(\epsilon, T(p_k[n+1])))$ and $P_k[n] \rightarrow P$

$\lim f(z) = p$. It is possible that y is not a simple arc, but by³ we can replace y by a simple arc $y' \subset y$. Thus $y \in X$, and we have shown that $Y \subset X$.

Suppose $x \in X$. Let y_0 be an arc at x such that f approaches a limit p^{-1} along y_0 . Take any n . Choose k with $p' \in p_k$.

Choose m so that x is in the

interior of $Q(n, m) \cap C$. Then y_0 has a subarc y_{jj} , with one endpoint at x , such that

$$t'q - \{x\} \subset Q(n, m) \cap f^{-1}(X \setminus p_k) \cap j.$$

Hence, for some f , $x \in \text{acc}[U(n, m, k, \mathcal{L})] = A(n, m, k, f)$. This shows that

$$\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{f=1}^{\infty} A(n, m, k, \mathcal{L}).$$

$$n=1 \quad m=1 \quad k=1 \quad f=1$$

By Lemma 2, the set

$A(n, m, k, J) - I_0(U(n, m, k, \mathcal{L})) = A(n, m, k, \mathcal{L}) - [I_0(U(n, m, k, \mathcal{L})) \cap A(n, m, k, \mathcal{L})]$ is countable. It follows by a routine argument that

$$A \cup A(n, m, k, j) \subset A \cup [I_0(U(n, m, k, \mathcal{L})) \cap A(n, m, k, I)] \cap n_{m,k,\mathcal{L}} \cap n_{m,k,\mathcal{L}}$$

is countable. Because

$$A \cup [I_Q(U(n, m, k, \mathcal{L})) \cap A(n, m, k, \mathcal{L})] = Y \subset X \subset A \cup A(n, m, k, \mathcal{L}), \quad n_{m,k,j} \cap n_{m,k,f}$$

the set $X - Y$ is countable, and therefore X is of type F_a . \square

Before stating our generalization of the foregoing theorem, we must say a few words about spaces of closed sets. If K is a bounded metric space with metric p , let $\hat{\mathcal{K}}(K)$ denote the set of all nonempty closed subsets of K . Hausdorff [1, page 146] defined a metric p on $\hat{\mathcal{K}}(K)$ by setting

$$p(A, B) = \max\{\sup_{a \in A} \text{dist}(a, B), \sup_{b \in B} \text{dist}(b, A)\},$$

$$a \in A \quad b \in B$$

where $\text{dist}(x, E)$ denotes $\inf_{e \in E} p(x, e)$. If K is compact, then $\hat{\mathcal{K}}(K)$ is a compact

$$e \in E$$

metric space with p as metric [1, page 150].

If f maps D into K and if y is an arc at a point $x \in C$, we let $C(f, y)$ denote the cluster set of f along y ; that is, we write

³ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*,

$C(f, y) = \{p \in K \mid \text{there exists a sequence } \{z_n\} \subset y \text{ AD such that } z_n \rightarrow x \text{ and } f(z_n) \rightarrow p\}.$

THEOREM 2. *Let K be a compact metric space, and let δ be a closed subset of $\hat{\alpha}(K)$. Let $f: D \rightarrow K$ be a continuous function. Then*

$\{x \in c \mid \text{there exists an arc } y \text{ at } x \text{ and there exists}$

$E \in \delta \text{ such that } C(f, y) \subset E\}$

is a set of type F_δ .

Proof. If $\delta > 0$ and $E \in \hat{\alpha}(K)$, let

$Z_\delta(\delta, E) = \{a \in K \mid \text{there exists } b \in E \text{ with } p(a, b) < \delta\}.$

Note that $Z_\delta(\delta, E)$ is open and that

$F \in \hat{\alpha}(K), p(E, F) < \delta \Rightarrow F \subset Z_\delta(\delta, E).$

Let $\{P(k)\}_{k=1}^\infty$ be a countable dense subset of δ (such a subset exists, because every compact metric space is separable). Let

$X = \{x \in c \mid \text{there exist an arc } y \text{ at } x \text{ and an } E \in \delta$

$\text{such that } C(f, y) \subset E\}.$

\mathbb{R}^n

Let $\{Q(n, m)\}_{m=1}^\infty$ be defined as in the proof of the preceding theorem. Let $\{U(n, m, k, \ell)\}_{\ell=1}^\infty$ be a counting (with repetitions allowed) of the components of $\bigcup_{p < k} Q(n, m, p, \ell)$.

Let $A(n, m, k, \ell) = \text{acc}[U(n, m, k, \ell)]$, and set

CO CO CO OO

$Y = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \bigcup_{k=1}^\infty \bigcup_{\ell=1}^\infty I_0(U(n, m, k, \ell)) \cap A(n, m, k, \ell).$

$n = 1, m = 1, k = 1, \ell = 1$

Since $I_0(U(n, m, k, \ell))$ is open, it is of type F_σ . It follows that Y is of type F_σ .

I claim that $Y \subset X$. Take any $y \in Y$. For each n , choose $m[n], k[n], \ell[n]$ so that

(3) $y \in I_0(U(n, m[n], k[n], \ell[n])) \cap A(n, m[n], k[n], \ell[n]).$

Set $U_n = U(n, m[n], k[n], \ell[n])$. Since δ is compact, there exist a $P \in \delta$ and some strictly ascending sequence $\{n_i\}_{i=1}^\infty$ of natural numbers such that

$\bigcap_{j=1}^\infty U_{n_j} \neq \emptyset$

$P(k[n_j]) \cap P.$

By (3) and Lemma 3, U_{n_j} and U_n have some point Z_j in common. For each j , choose an arc $y_j \subset U_{n_j}$ with one endpoint at Z_j and the other at Z_{j+1} . Then

$y_j \subset Q(n_{j+1}, m[n_{j+1}])$. Also,

$y \in A(n_{j+1}, m[n_{j+1}], k[n_{j+1}], \ell[n_{j+1}]) \subset U \subset Q(n_{j+1}, m[n_{j+1}]),$

and therefore each point of y has distance less than $\frac{1}{n_j}$ from y . Now

J H j + 1

$2t + 1 \mid \ell$

0 as $j \rightarrow \infty$ therefore, if we set $y = \{y_j\} \cup \bigcup_{i=1}^\infty U_i$, then y is an arc with $n_j + 1$

one endpoint at y .

I claim that $C(f, y) \subset P$. Take any $p \in C(f, y)$. There exists a sequence $\{w_s\}_{s=1}^\infty$ in $y - \{y\}$ such that $w_s \rightarrow y$ and $f(w_s) \rightarrow p$. Let ϵ be an arbitrary positive number. Choose j_0 so that $p(P(k[n_j]), P) < \epsilon/3$ for all $j > j_0$. Choose so that $j > j_0$ implies $1/n_{j+1} < \epsilon/3$. We can choose an s such that $w_s \in y^\wedge$ for some $i > j_0$, j and such that

$$(4) \quad p(f(w_s), p) < \epsilon/3$$

•

Then

$$f(w_s) \in f(y) \subset f(U_{n \cdot H}) \subset P(k[n_{i+1}]),$$

and therefore we can choose a point $q \in P(k[n_{i+1}])$ with

$$(5)$$

$$P(f(w_s), q) < \epsilon/3 \quad \square$$

E

3*

Moreover, because $p(P(k[n_{i+1}]), P) < \epsilon/3$, there exists some $q' \in P$ with

$$(6)$$

$$p(q, q') < \epsilon/3$$

Together, (4), (5), and (6) show that $p(p, q') < \epsilon$. Since P is closed and ϵ is arbitrary, this proves that $p \in P$. Hence $C(f, y) \subset P$ (8). By⁴, we can if necessary replace y by a simple arc $y^1 \subset y$; it follows that $y \in X$. Thus $Y \subset X$.

Now suppose $x \in X$. Choose an arc y_0 at x such that $C(f, y_0) \subset P_0$ for some $P_0 \in \mathcal{P}$. Take any n . Choose k with $p(P_0, P(k)) < 1/n$. Then

$$p_0 \in C(f, y_0) \subset P(k)$$

$$\text{hence } C(f, y_0) \subset P(k).$$

Choose m so that x is in the interior of $Q(n, m) \cap C$.

If for each natural number t there exists a point $z'_t \in y_0 \cap S \times j \cap AD$ with $p(z'_t) \rightarrow p(k)$, then $f(z'_t) \in K \cap S \cap (P(k) \cap j)$,

and since $K \cap S \cap (P(k) \cap j)$ is compact, there exist some $a \in K \cap S \cap P(k) \cap j$ and a subsequence $\{f(z'_t)\}_{t \in \mathbb{N}}$ such that $f(z'_t) \rightarrow a$. But then $a \in C(f, y_0)$, contrary to (8). \square

the relation $C(f, y_0) \subset P(k)$. We conclude that there exists a natural number t for which

$$y_0 \cap S \cap (P(k) \cap j) \neq \emptyset$$

It follows that y_0 has a subarc y^\wedge with one endpoint at x such that

$$y^\wedge \cap \{x\} \subset P(k) \cap j \cap Q(n, m).$$

Hence there exists an ϵ such that

$$x \in \text{acc } [U(n, m, k, \epsilon)] = A(n, m, k, \epsilon).$$

This shows that $\text{co } \infty \text{ co } \text{co } x \in U \cup A(n, m, k, \epsilon)$. $n=1 \quad m=1 \quad k=1 \quad \epsilon=1$

⁴ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

By Lemma 2, the set

$A(n, m, k, \mathcal{L}) - I_0(U(n, m, k, \mathcal{L})) = A(n, m, k, J?) - [I_0(U(n, m, k, \mathcal{L})) \cup A(n, m, k, \mathcal{L})]$ is countable. It follows easily that

$A \cup A(n, m, k, \mathcal{L}) - A \cup [I_0(U(n, m, k, \mathcal{L})) \cup A(n, m, k, \mathcal{L})] \cap \mathcal{M}_{m,k,\mathcal{L}} \cap \mathcal{M}_{m,k,\mathcal{L}}$ is countable. Since

$A \cup [I_0(U(n, m, k, \mathcal{L})) \cup A(n, m, k, 4)] = Y \subset X \subset A \cup A(n, m, k, \mathcal{L}), \mathcal{M}_{m,k,\mathcal{L}} \cap \mathcal{M}_{m,k,\mathcal{L}}$

$X - Y$ must be countable. Thus X is the union of an F_a -set and a countable set, and hence it is of type F_a . \square

In each of the following four corollaries, let f denote a continuous function mapping D into the Riemann sphere.

COROLLARY 1 (J. E. McMillan). *Let E be a closed subset of the Riemann sphere. Then the set*

$\{x \in C \mid \text{there exist an arc } y \text{ at } x \text{ and a point } p \in E \text{ such that } \lim_{z \rightarrow x, z \in y} f(z) = p\}$

is of type F_{a6} .

COROLLARY 2. *Suppose $d > 0$. Then the set*

$\{x \in C \mid \text{there exists an arc } y \text{ at } x \text{ such that } [\text{diameter } C(f, y)] < d\}$

is of type F_{Q6} .

COROLLARY 3. *Let E be a closed subset of the Riemann sphere. Then the set*

$\{x \in C \mid \text{there exists an arc } y \text{ at } x \text{ with } C(f, y) \subset E\}$

is of type F_{a6} .

COROLLARY 4. *The set*

$\{x \in C \mid \text{there exists an arc } y \text{ at } x \text{ such that } C(f, y) \text{ is an arc of a great circle}\}$

is of type F_{Q6} .

We can obtain all these corollaries by taking \mathcal{S} to be a suitable family of closed sets and applying Theorem 2. To prove Corollary 4, we need the fact that $C(f, y)$ is always connected. One could go on listing such corollaries ad infinitum, but we refrain.

It is interesting to note that in Corollary 1 it is not necessary to assume that E is closed. By combining Corollary 1 with Theorem 6 of⁵, one can prove that the conclusion of Corollary 1 holds even if E is merely assumed to be of type F_a .

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⁵ F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math*,

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BOUNDARY-FUNCTIONS

by

Theodore John Kaczynski

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BOUNDARY FUNCTIONS

By Theodore John Kacijnski

Abstract

Let H denote the set of all points in the Euclidean plane having positive y -coordinate, and let X denote the x -axis. If p is a point of X , then by an arc at p we mean a simple arc v , having one endpoint at p , such that $v - \{p\} \subset H$. Let f be a function mapping H into the Riemann sphere. By a boundary function for f we mean a function t defined on a set $E \subset X$ such that for each $p \in E$ there exists an arc v at p for which

$$\lim_{z \rightarrow p} f(z) = t(p).$$

$$z \rightarrow p$$

$$z \in V$$

The set of curvilinear convergence of f is the largest set on which a boundary function for f can be defined; in other words, it is the set of all points $p \in X$ such that there exists an arc at p along which f approaches a limit. A theorem of J.E. McMillan states that if f is a continuous function mapping H into the Riemann sphere, then the set of curvilinear convergence of F is of type $F(sd)$. In the first of two chapters of this dissertation we give a more direct proof of this result than McMillan's, and we prove, conversely, that if A is a set of type $F(sd)$ in X , then there exists a bounded continuous complex-valued function in H having A as its set of curvilinear convergence. Next, we prove that a boundary function for a continuous function can always be made into a function of Baire class 1 by changing its values on a countable set of points. Conversely, we show that if t is a function mapping a set $E \subset X$ into the Riemann sphere, and if t can be made into a function of Baire class 1 by changing its values on a countable set, then there exists a continuous function in H having t as a boundary function. (This is a slight generalization of a theorem of Bagemihl and Piranian.) In the second chapter we prove that a boundary function for a function of Baire class $e > 1$ in H is of Baire class at most $e + 1$. It follows from this that a boundary function for a Borel-measurable function is always Borel-measurable, but we show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable. The dissertation concludes with a list of problems remaining to be solved.

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INTRODUCTION

1. Preliminary Remarks

Let H denote the upper half-plane, and let X denote its frontier, the x -axis. If $x \in X$, then by an arc at x we mean a simple / —————

arc y with one endpoint at x such that y - Suppose that f

is a function mapping H into some metric space Y . If E is any subset

I of X , we will say that a function $\langle p: E \rightarrow Y \rangle$ is a *boundary function* for f if, and only if, for each $x \in E$ there exists an arc Y at x such that

$$\lim_{z \rightarrow x} f(z) = \langle p \rangle(x) \text{ as } z \in Y$$

The study of boundary functions in this degree of generality was initiated by Bagemihl and Piranian¹. A function defined in H may have more than one boundary function defined on a given set $E \subset X$, but it follows from a famous theorem of Bagemihl² that any two such boundary functions differ on at most a countable set of points.

If f is defined in H , then the *set of curvilinear convergence* of f is the set of all points $x \in X$ such that there exists some arc α at x along which f approaches a limit. Evidently, this is the largest set on which a boundary function for f can be defined.

J. E. McMillan [10] discovered that the set of curvilinear convergence of a continuous function is always of type F and in this paper we show that every set of type F in X is the set of curvilinear convergence of some continuous function. Next, we show that if f is a function defined on a subset E of X , then $\langle p \rangle$ is a boundary function for some continuous function if and only if $f|_E$ can be made into a function of the first Baire class by changing its values on at most a countable set of points. (This solves a problem of Bagemihl and Piranian [2, Problem 1].) We then consider functions that are not assumed to be continuous, and we prove that a boundary function for a function of Baire class n is of Baire class at most $n+1$ (thus proving another conjecture of Bagemihl and Piranian³). It follows from this that a boundary function for a Borel-measurable function is always Borel-measurable, and in the last section we show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable.

Most of the results appearing here have already been published ([8] and [9]). At the time I published these papers I did not expect to have to make use of this material for a dissertation.

¹ F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

² F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci. U. S. A.* 4 (1955) 379-382.

³ F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

2. Notation

\mathbb{R} will denote the field of real numbers, \mathbb{R}^n

\mathbb{R}^n will denote n -dimensional Euclidean space.

Points in \mathbb{R}^n will be written in the form $\{x_1, \dots, x_n\}$ rather than (x_1, \dots, x_n) (to avoid confusion with open intervals of real numbers in the case $n = 2$).

If $v \in \mathbb{R}^n$, then $|v|$ denotes the length of the vector v .

S^n denotes $\{v \in \mathbb{R}^n : |v| = 1\}$. S^2 will be referred to as the *Riemann sphere*.

. Let

$$H = \{ \langle x, y \rangle \in \mathbb{R}^2 : y > 0 \}$$

$$H_n = \{ \langle x, y \rangle \in \mathbb{R}^2 : y > 0 \} \quad X = \{ \langle x, 0 \rangle : x \in \mathbb{R} \} \quad X_n = \{ \langle x, 0 \rangle : x \in \mathbb{R} \} \bullet$$

We consider X as being identical with \mathbb{R} . Thus, for example, $\langle x, 0 \rangle < \langle x', 0 \rangle$ means $x < x'$, and for $p, q \in X$, the notations $[p, q]$, (p, q) , etc. refer to the obvious intervals on X .

If E is a subset of a topological space, then E'' denotes the interior of E .

\bar{E} denotes the closure of E , E° denotes the interior of E , and E^c denotes the complement of E .

Of course, if E is a subset of X , then E means the interior of E relative to X , not relative to the whole plane. In Section 7, we often denote two line segments by s and s' . Since the prime notation is never used for complementation in that section, there is no danger of confusing s' with the complement of s .

2

If f is a function defined in a subset of \mathbb{R}^2 , then $f(x, y)$ means $f(\langle x, y \rangle)$. Thus we write $f(z)$ for $z \in \mathbb{R}$ and $f(x, y)$ for $x, y \in \mathbb{R}$ interchangeably.

3. Baire Functions

In this section we review the main facts concerning Borel sets and Baire functions, and we prove some results that will be needed later.

If \mathcal{C} is any family of sets, let \mathcal{C}_g be the family of all sets that can be written as a countable intersection of members of \mathcal{C} , and let \mathcal{C}_u be the family of all sets that can be written as a countable union of members of \mathcal{C} .

Suppose M is a metrizable topological space. Let $\mathcal{P}(M)$ be the family of all open subsets of M and let $\mathcal{Q}(M)$ be the family of all closed subsets of M . If α is an ordinal number greater than 1, let

$$\mathcal{P}^\alpha(M) = (\bigcap \mathcal{Q}^\beta(M))_{\beta < \alpha}$$

$$\mathcal{Q}^\alpha(M) = (\bigcup \mathcal{P}^\beta(M))_{\beta < \alpha}$$

For any $C, E \in \mathcal{Q}^\alpha(M)$ $C \leq E$ iff $C \subseteq E$.

For any subset L of M , $E \in \mathcal{P}^\alpha(L)$ (respectively $\mathcal{Q}^\alpha(L)$) if and only if there exists a set $D \in \mathcal{P}^\alpha(M)$ (respectively $\mathcal{Q}^\alpha(M)$) such that $e = d \cap L$.

$\mathcal{P}^\alpha(M)$ and $\mathcal{Q}^\alpha(M)$ are closed under finite unions and finite intersections. $\mathcal{P}^\alpha(M)$ is closed under countable unions and $\mathcal{Q}^\alpha(M)$ is closed under countable intersections.

If $\alpha < \beta$, then $\mathcal{P}^\alpha(M) \cup \mathcal{Q}^\alpha(M) \subseteq \mathcal{P}^\beta(M) \cup \mathcal{Q}^\beta(M)$.

Let $F^\circ(M)$ be the class of all F° sets of M , and let $G^\circ(M)$ be the class of all G° sets of M .

2 2

$P(M) = F(M)$ and $Q(M)' = G^\circ(M)$.

Let Y be a metric space. For any family C of subsets of M we will say that a function $f : M \rightarrow Y$ is of *class* (C) if and only if $f^{-1}(U) \in C$ for every open set $U \subset Y$.

The following definition of the Baire classes is somewhat different from the classical definition, but it seems more convenient for our purposes. A function $f : M \rightarrow Y$ is said to be of *Baire class* $0(M, Y)$ if and only if it is continuous. If α is an ordinal number greater than or equal to 1, then f is said to be of *Baire class* α if and only if there exists a sequence of functions mapping M into Y , $\{f_n\}$ being of Baire class $n_n(M, Y)$ for some $n_n < \alpha$, such that $f = \bigcup_{n=1}^{\infty} f_n$ pointwise.

If $f : M \rightarrow Y$ is of Baire class $\alpha(M, Y)$ and if L is a subset of M , then $f|_L$ is of Baire class $\alpha(L, Y)$.

If K is a metric space, if $g : K \rightarrow M$ is continuous, and if $f : M \rightarrow Y$ is of Baire class $\alpha(M, Y)$, then the composite function $f \circ g$ is of Baire class $\alpha(K, Y)$.

If Y is separable and if $f : M \rightarrow Y$ is of Baire class $\alpha(M, Y)$, then f is of class $(P^{\alpha+})$ [4, page 294].

If Y is separable and arcwise connected, if $g \in P^1$, and if $f : M \rightarrow Y$ is of class $(P^{\alpha+})$, then f is of Baire class $\alpha(M, Y)$ ⁵.

For any α , if $f : M \rightarrow R$ is of class $(P^{\alpha+})$, then f is of Baire class $\alpha(M, R)$ ⁶.

If $f \in P^{\alpha+}$ and $f : L \rightarrow R$ is of Baire class $\alpha(L, R)$, then f can be extended to a function $f : M \rightarrow R$ of Baire class $\alpha(M, R)$ ⁷.

We say that a function $f : M \rightarrow R$ is *Borel measurable* if, and only if, for every open set $U \subset R$, $f^{-1}(U)$ is a member of the σ -ring of subsets of M generated by the open sets.

If $f : M \rightarrow R$ is of some Baire class $\alpha \in C^M$, then f is Borel-measurable, and, conversely, if $f : M \rightarrow R$ is Borel-measurable, then f is of Baire class $\alpha(M, R)$ for some countable ordinal number α [7, page 294].

The proofs of Lemmas 1 through 6 are based on standard techniques in the study of Baire functions.

Lemma 1. Let M be a metric space, and let E and F be two F° sets in M . Then there exist two disjoint F° sets A and $B \subset M$ such that

$E - F \subset A$ and $F - E \subset B$.

\square

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$, where E_n and F_n are closed, $n \in \mathbb{N}$.

⁵ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

⁶ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

⁷ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

Then

$$B_n, F_n \in \mathcal{F}_0(M) \text{ a } g_6(M).$$

It is easy to check that $F \wedge fM) \wedge G \wedge fM)$ is an algebra (i.e., is closed under complementation, finite unions, and finite intersections). We inductively define a sequence of pairs of sets (A_n, B_n) as follows. Let

$$A_x = E_x, B_x = F_x \text{ A } A'.$$

For $n > 1$, let

$$n-1 \text{ n}$$

$$A = E \text{ n } C \setminus B!, B = F \text{ n } A! \text{ . n n}^n \text{ ' n n}'' \text{ j}$$

By induction, $A_n, B_n \in \mathcal{F}(M) \wedge G(M)$. Let , **CO 00**

$$A = \bigcup_{n=1}^{\infty} A_n, B = \bigcup_{n=1}^{\infty} B_n.$$

$$n=1 \text{ n}=1$$

Then A and B are \mathcal{F} sets. Notice that

$$(j^B - \mathcal{F}^F \text{ O } A. g e ,$$

$$j=1^3 \text{ j}=1^3$$

from which it follows that $n-1$,

$$A = E \text{ a } (\bigcup B.) \text{ O } E \text{ A } F', \text{ n n}^{z v} \text{ — n}^*$$

$$1=1^J$$

and n ,

$$B_n \boxtimes y \text{ 2 } P_{n^{AB}}, \bullet$$

$$3=1.$$

Therefore

$$00$$

$$A \text{ 5 LJ CE}_n \text{ A } F') = E - F \text{ n}=1$$

and

$$\text{CO}$$

$$B \text{ “ } \bigcup C F_n A E') = F - E. \text{ n}=1$$

It only remains to show that $A \wedge B = \langle j \rangle$. Suppose $x \in A \wedge B$. Choose

\mathcal{L}, m with $x \in \mathcal{L} \wedge A$ and $x \in B$. If $\mathcal{L} > m$, then $\mathcal{L} > 1$, so that

$J \in$ in

Hence $x \in B'$ —a contradiction. On the other hand, if $St, < m$, then $m \text{ —}$

m

$$f.n \text{ A } A! \text{ a a; , } 1=1^J$$

so that $x \in \mathcal{L} \wedge A$ } —another contradiction. We conclude that $A \wedge B =$

If E is a subset of a space M , we let x_E denote the characteristic function of E .

Lemma 2. Let L be a. subset of a metric space M , and suppose that

co

$E \wedge G \wedge F \wedge CL) / A \wedge G \wedge fL)$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous real-valued functions on M such that $f \wedge \rightarrow x_E$ pointwise on L .

Proof. Both E and $L - E$ are in $F \wedge CL)$, so there exist sets $E^\wedge, F^\wedge \in \mathcal{F} \wedge fM)$ such that

$$E = E^\wedge \wedge L \text{ and } L - E = F^\wedge \wedge L.$$

By Lemma 1, there exist $A, B \in \mathcal{F}(M)$ such that $A \wedge B = \langle \rangle$ and

$E \wedge F \leq G \wedge A, F \wedge E \leq G \wedge B$. We have

$E = A \wedge L$ and $L \wedge E = B \wedge L$.

$0, 0$.

Write $A = \bigcup_{n=1}^{\infty} A_n$ where A_n are closed and $A_n \cap A_{n+1} = \emptyset$

$B \wedge B$ for each n . By Urysohn's Lemma there exists a continuous function $f_n : M \rightarrow [0, 1]$ such that

$f_n(x) = 1$ when $x \in A_n$

$f_n(x) = 0$ when $x \in A_{n+1}$

$f_n(x) = 0$ when $x \in B$.

$n \in \mathbb{N}$

\mathbb{C}^0

$\{f_n\}_{n=1}^{\infty}$ is the desired sequence. \square

Lemma 3. Let L be a subset of a metric space M , $f : L \rightarrow \mathbb{R}$ a function of class $(F_a(L))$ that takes only finitely many different values.

Then there exists a sequence $\{f_n\}$ of continuous real-valued functions on M such that $f_n \rightarrow f$ pointwise on L .

Proof. From Banach's Hilfssatz 3⁸, we see that there exist real numbers a_1, \dots, a_n and sets

$1 \leq q \leq n$

$E_1, \dots, E_q \in \mathcal{F}_0(L) \cap \mathcal{G}_6(L)$

such that

$1 \leq q \leq n$

$f = \sum_{i=1}^n a_i \chi_{E_i}$

$1 \leq i \leq n$

$j=1, \dots, J$

\square \square

If we choose for each j a sequence $\{f_{n,j}\}$ of continuous real-valued

functions on M such that $f_{n,j} \rightarrow f$ pointwise on L , and if we set $n_j = n$

\square \square

then is the desired sequence. \square

Lemma- 4. Let L be a metric space, f a bounded real-valued function

\bullet \bullet \mathbb{C}^0

on L of Baire class 1(L, \mathbb{R}). Then there exists a sequence $\{f_n\}$ of real-valued functions on L converging uniformly to f , such that each f is of class $(F \wedge fL)$ and takes only finitely many different values. '

Proof, f is of class $(F \wedge CL)$ and the range of f is totally bounded, so an obvious modification of the proof of Banach's Hilfssatz 4⁹ gives the desired result. \square

⁸ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

⁹ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

Lemma 5. Let M be a metric space, L a subset of M , $f : L \rightarrow \mathbb{R}$ a function of Baire class 1 (L, R). Then there exists a sequence

of continuous real-valued functions on M such that $f = \lim_{n \rightarrow \infty} f_n$ pointwise on L .

Proof. We first* prove the lemma under the assumption that f is bounded. For any bounded real-valued function h , let •

$U_h = \{x \in L : |h(x)| < \epsilon\}$.

By Lemma 4 we can choose, for each n , a function $g_n : L \rightarrow \mathbb{R}$ of class $(F \cap C)$ such that g_n takes only finitely many different values and

Let $h_n = g_n$ for $n \geq 1$.

Then, for $\epsilon > 0$,

$U = \bigcup_{n=1}^{\infty} U_n$.

Each h_n

values,

is of class $(F \cap C)$ and takes only finitely many different

* * 1 00

so by Lemma 3 we can choose (for each n) a sequence $\{h_n^j\}_{j=1}^{\infty}$

of continuous functions on M such that $h_n^j \rightarrow h_n$ pointwise on L .

Set

$k_n^j(x) = |h_n^j(x) - h_n(x)|$ if $h_n^j(x) < \epsilon_n$ and $|h_n^j(x) - h_n(x)| > \epsilon_n$

$k_n^j = \epsilon_n$ if $h_n^j(x) > \epsilon_n$

$k_n^j = \epsilon_n$ if $h_n^j(x) < \epsilon_n$ and $|h_n^j(x) - h_n(x)| > \epsilon_n$

Then k_n^j is continuous, $k_n^j \rightarrow 0$ pointwise on L , and $\sum_{n=1}^{\infty} k_n^j < \infty$. Therefore, if we set $2^{n+2} = 4^n$

$f_n = \sum_{j=1}^n k_j^j$,

$\sum_{j=1}^n k_j^j$

$\sum_{j=1}^n k_j^j$

then the series converges uniformly and f_n is continuous on M . We claim that $f_n \rightarrow f$ pointwise on L . Take any $x \in L$ and any $\epsilon > 0$. Choose m large enough so that $\sum_{n=m+1}^{\infty} 4^{-n} < \epsilon/4$

* For each $n \geq m$ choose $j(n)$ so $2^{j(n)} > 4^n$ that

$j(n) - M \leq j(n) \leq M$

Let $i = \max \{j(1), \dots, j(m)\}$. Then $j > i$ implies that

$|f(x) - f_n(x)| < |f(x) - E_{k^{j(n)}}(x)| + \sum_{n=1}^{\infty} 4^{-n}$

$\sum_{n=1}^{\infty} 4^{-n}$

$\sum_{n=1}^{\infty} 4^{-n}$

$\sum_{n=1}^{\infty} 4^{-n}$

$\sum_{n=1}^{\infty} 4^{-n} + \sum_{n=m+1}^{\infty} 4^{-n} < \epsilon$

$\sum_{n=1}^{\infty} 4^{-n}$

$\sum_{n=1}^{\infty} 4^{-n} + \sum_{n=m+1}^{\infty} 4^{-n} < \epsilon$

$\sum_{n=1}^{\infty} 4^{-n} + \sum_{n=m+1}^{\infty} 4^{-n} < \epsilon$

$\sum_{n=1}^{\infty} 2^{-n}$

Thus $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for each $x \in L$, and the lemma

$\sum_{n=1}^{\infty} |f_n(x)| < \infty$

$\sum_{n=1}^{\infty} |f_n(x)| < \infty$

$\sum_{n=1}^{\infty} |f_n(x)| < \infty$

is proved for bounded $\sum_{n=1}^{\infty} f_n$.

If f is not bounded, let

$g(x) = \arctan f(x)$

Then $-\pi/2 < g(x) < \pi/2$ for every $x \in L$, and g is of Baire class 1(L, \mathbb{R}),

so

there exists a sequence $\{g_n\}$ of continuous functions on M converging to g pointwise on L . Set

$h_n(x) = g_n(x) - g(x)$

$h_n(x) = g_n(x) - g(x)$

$h_n(x) = g_n(x) - g(x)$

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Then h_n is continuous on M , $-\pi/2 < h_n(x) < \pi/2$ and $\sum_{n=1}^{\infty} h_n$ converges pointwise on

L . Let $f(x) = \tan \sum_{n=1}^{\infty} h_n(x)$. Then f is continuous on M and $f = \sum_{n=1}^{\infty} f_n$

pointwise on L .

Lemma 6. If L is a subset of a metric space M and $f : L \rightarrow \mathbb{R}^m$ is a function, then the following are equivalent.

(i) f is of Baire class 1(L, \mathbb{R}^m).

(ii) f is of class (F_σ) .

(iii) There exists a sequence $\{f_n\}$ of continuous functions mapping M into \mathbb{R}^m such that $f = \sum_{n=1}^{\infty} f_n$ pointwise on L .

This lemma is an easy consequence of Lemma 5.

3

Definition. Let q be any point of \mathbb{R}^n lying inside the bounded open

domain determined by S . By the q -*projection* of $\mathbb{R}^n - \{q\}$ onto S we

mean the function P^q defined as follows. If a is any point of

$\mathbb{R}^n - \{q\}$, let I be the unique ray, having its endpoint at q , that

passes through a , and let $P^q(a)$ be the intersection point of I with

S . P^q is a continuous mapping of $\mathbb{R}^n - \{q\}$ onto S that fixes every

point of S .

Theorem 1. Let L be an arbitrary subset of \mathbb{R}^n . Then a function

$f : L \rightarrow \mathbb{R}^m$ is of Baire class 1(L, \mathbb{R}^m) if and only if it is of class (F_σ) .

Proof. Assume that $f : L \rightarrow \mathbb{R}^m$ is of class (F_σ) . $S \subset \mathbb{R}^m$, so by

Lemma 6 there exists a sequence of continuous functions mapping

\mathbb{R}^n

into \mathbb{R}^m such that $f = \sum_{n=1}^{\infty} f_n$ pointwise on L . Let

$A_n = \{x \in \mathbb{R}^n : \sum_{k=1}^n |f_k(x)| < \infty\}$

11 11

$$B_n = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^3 : \|x\| \leq \frac{1}{k}\}$$

11 11

$$C_n = \bigcup_{k=1}^{\infty} \{x \in \mathbb{R}^3 : \|x\| \leq \frac{1}{k}\}$$

11 11 L^*

Let $f^0 = f_n|_{L^*}$. According to [5, Lemma 2.9, page 299], f^0 can be extended to a continuous function $g_n : \mathbb{R}^3 \rightarrow \{v \in \mathbb{R}^3 : |v| = 1\}$.

Define $h : \mathbb{R}^3 \rightarrow \mathbb{R} \setminus \{0\}$ by setting

$$h(x) = g_n(x) \text{ if } x \in B_n$$

$$h(x) = f_n(x) \text{ if } x \in C_n.$$

Since B_n are closed, h is continuous, and it is easy to verify that $h(x) \neq 0$ for each $x \in L$. Let $k : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the composite

function $P \circ h$. Then k is continuous, and for each $x \in L$, $0 < k(x) < \infty$.

2

$k(x) \rightarrow P_0(f(x)) = f(x)$. Thus f is of Baire class 1(L, \mathbb{R}).

Definition. Let M and Y be metric spaces. Then a function $f : M \rightarrow Y$

is said to be of honorary Baire class 2(M, Y) if and only if there exists a countable set $N \subset M$ and a function $g : M \setminus N \rightarrow Y$ of Baire class

1($M \setminus N, Y$) such that $f(x) = g(x)$ for every $x \in M \setminus N$.

2

Theorem 2. Let L be an arbitrary subset of \mathbb{R}^n and let Y be either

2

the real line, a finite-dimensional Euclidean space, or S^1 . Then a function $f : L \rightarrow Y$ is of honorary Baire class 2(L, Y) if and only if there exists a countable set $N \subset L$ such that $f|_{L \setminus N}$ is of class $(F_1(L \setminus N))$.

Proof. Suppose that $f : L \rightarrow Y$ is of honorary Baire class 2(L, Y). Then there exists $g : L \setminus N \rightarrow Y$ of Baire class 1($L \setminus N, Y$) and a countable set $N \subset L$ such that $f|_{L \setminus N} = g|_{L \setminus N}$. But $g|_{L \setminus N}$ is of class $(F_1(L \setminus N))$. Conversely, suppose that $f|_{L \setminus N}$ is of class $(F_1(L \setminus N))$, where N is countable. We must show that f is of honorary Baire class 2(L, Y). First consider the case where $Y = \mathbb{R}^m$. Write

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)).$$

Then $f_i|_{L \setminus N}$ is of class $(F_1(L \setminus N))$ ($i=1, \dots, m$), and it follows that $f|_{L \setminus N}$ is of Baire class 1($L \setminus N, \mathbb{R}^m$). Since $L \setminus N \in G_\delta(L)$, we can extend $f|_{L \setminus N}$ to a function $g : L \rightarrow \mathbb{R}^m$ of Baire class 1(L, \mathbb{R}^m). If we set $g(x) = (g_1(x), \dots, g_m(x))$, then g is of Baire class 1(L, \mathbb{R}^m) and $g(x) = f(x)$ for $x \in L \setminus N$, so we have the desired result.

2 2 3

Now consider the case where $Y = S^1$. Since $S^1 \subset \mathbb{R}^2$, there exists, as we have just shown, a function $g : L \setminus N \rightarrow \mathbb{R}^2$ of Baire class 1($L \setminus N, \mathbb{R}^2$) such that $g(x) = f(x)$ for all $x \in L \setminus N$. Then $g(L \setminus N) \subset S^1$ is countable, so

there exists some point q in the bounded open domain determined by S^1 such that $q \notin g(L \setminus N)$. Let h be the

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$P \circ g$. Then h maps L into S^1 , and for each $x \in L \setminus N$

$h(x)$

$P(gM) = P(f(x))$

composite function “ $N, f(x)$.

2

If $(JC\ S$ is open, then

$h^{-1}(U)$

$^{-1}(P \text{ } ^\wedge U)) \in F\ CL)$,

so h is of class $(F\ (L))$. By Theorem 1, h is of Baire class

2 M

$1(L, S)$, so we have the desired result. \textcircled{R}

CHAPTER I. BOUNDARY FUNCTIONS FOR CONTINUOUS FUNCTIONS

If r is a positive number and if y_0 is a point of a metric space Y having metric p , then

$S(r, Y_0)$ denotes ‘ $\{y \in Y : p(y, y_0) < r\}$.

We will repeatedly make use of Theorem 11.8 on page 119 in [11] without making explicit reference to it. This theorem states that if D is a Jordan domain in R or in $R \cup \{\infty\}$, if y is the frontier of D , and if a is a cross-cut in D whose endpoints divide y into arcs y_1 and y_2 , then $D-a$ has two components, and the frontiers of these components are respectively $a \cup y_1$ and $a \cup y_2$ (The term cross-cut is defined on page 118 in [11].)

4. Domain of the Boundary Function

Definition. If f is a function mapping into a metric space Y , then ‘ the *set of curvilinear convergence* of f is defined to be

$\{x \in X : \text{there exists an arc } y \text{ at } x \text{ and there exists } y \in Y \text{ such that } \lim_{z \rightarrow x \text{ along } y} f(z) = y\}$.

J. E. McMillan [10] proved that for suitable spaces Y , the set of curvilinear convergence of a continuous function is always of type F_σ . We give a more direct proof of this result than McMillan’s.

(This proof can be modified to give a more general result; see [9].)

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11.16 \square

* An interval of X will be called *nondegenerate* if and only if it contains more than one point. \square

Suppose y is a cross-cut of H . If V is the bounded component

of $H - y$, let $L(y) = V \cap X$. Then $L(y) = [c, d]$, where c and d are the endpoints of y and $c < d$. Suppose Q is a domain contained in H . Let r denote the family of all cross-cuts y of H for which $y \cap Q \neq \emptyset$, and let

$$I(Q) = \bigcup_{y \in r} L(y)^*$$

Let $\text{acc}(Q)$ denote the set of all points on X that are accessible by arcs in Q .

Lemma 7. Assume that $\text{acc}(Q)$ is nonempty. Let a be the infimum of $\text{acc}(Q)$ and let b be the supremum of $\text{acc}(Q)$. Then

$$I(Q) = (a, b).$$

Proof. Suppose $x \in I(Q)$. Let y be a cross-cut of H such that $x \in L(y)$.

$y \cap H \cap Q \neq \emptyset$ and $x \in L(y) \subset [c, d]$, where c and d are the endpoints of y and $c < d$. It is evident that c and d are in $\text{acc}(Q)$, so $a \leq c < x < d \leq b$, and $x \in (a, b)$. Conversely, suppose $x \in (a, b)$. Then there exist points $c', d' \in \text{acc}(Q)$ with $c' < x < d'$. Since Q is arcwise connected, it is easy to show that there exists a crosscut y' of H , with $y' \cap Q \neq \emptyset$, having c', d' as its endpoints. But then $x \in (c', d') = L(y')$, so $x \in I(Q)$. \square

Lemma 8. If Q_1 and Q_2 are domains contained in H , and if $X \cap M$

$$(1) I(Q_1) \cap \text{acc}(Q_2) \neq \emptyset \text{ and } I(Q_2) \cap \text{acc}(Q_1) \neq \emptyset$$

are not disjoint, then Q_1 and Q_2 are not disjoint.

Proof. We assume that Q_1 and Q_2 are disjoint and derive a contradiction.

Let a be a point in both of the two sets (1). Let y be a cross-cut of H , with $a \in L(y)$ ($i = 1, 2$). Let

I_1 and A_1 be the components of $H - y$, where I_1 is the bounded component. Observe that $y \cap Q_1 \neq \emptyset$ and $y_2 \cap Q_2 \neq \emptyset$ are disjoint.

Suppose $y_1 \cap Q_1 \neq \emptyset$ and $y_2 \cap Q_2 \neq \emptyset$. Then, since $y \cap Q_1 \neq \emptyset$ has a point in common with Q_2 . But, since U is unbounded, U cannot be contained in Q_2 , so must have a point in common with $Q_2 \cap H$. This contradicts the assumption that $Q_1 \cap Q_2 = \emptyset$ so we conclude that either $y \cap Q_1 \neq \emptyset$ or $y_2 \cap Q_2 \neq \emptyset$. Hence, either $y \cap Q_1 \neq \emptyset$ or $y_2 \cap Q_2 \neq \emptyset$. By symmetry, we may assume that

$$y_2 \cap Q_2 \neq \emptyset$$

Q_2 does not meet y , and Q_2 does meet U (because $y_2 \cap Q_2 \neq \emptyset$ and $U \cap Q_2 \neq \emptyset$) so $Q_2 \subset U$. Since $a \in \text{acc}(Q_2)$, there exists a point $b \in L(y)$ such that $b \in \text{acc}(Q_2)$. But then $b \in Q_2 \cap U$ and this is impossible because the frontier of U is disjoint from $L(y)$. \square Theorem 3 (J. E. McMillan). Let Y be a complete separable metric space and let $f : H \rightarrow Y$ be a continuous function. Then the set of curvilinear convergence of f is of type F^* .

Proof. Let $\{p_v\}_v$ be a countable dense subset of Y . Let $\{Q(n, m)\}_{n, m=1}^\infty$

be a counting of all sets of the form

$$\{ \langle x, y \rangle : 0 < y < r \text{ and } r < t < r + i \}$$

where r is a rational number. Let $\{U(n, m, k, \ell)\}_{n, m, k, \ell=1}^\infty$ be a counting

(with repetitions allowed) of the components of

$$H - \bigcup_{k=1}^\infty \bigcup_{m=1}^\infty U(n, m, k, \ell)$$

$$Q(n, m).$$

(We consider $|$ to be a component of $\langle j \rangle$.) Let
 $A(n, m, k, \&) = \text{acc}[U(n, m, k, \&)]$.
Set
 $CO \ CO \ CO \bullet CO$
 $\gg \bullet n \ u \ u \ u \ I(U(n, m, k, \$)) \ n \ A(n, m, k, \&)$.
 $n=1 \ m=1 \ k=1 \ \&=1$
Since $I(U(n, m, k, \&))$ is open in X it is of type F . It follows that
 O'
 B is of type F . Let C denote the set of curvilinear convergence o 6
of f . I claim that $B \ \$C$. Take any $b \in B$. For each n , choose $m[n]$,
 $k[n]$, $\&[n]$ with I
(2) $b \in I(U(n, m[n], k[n], \&[n])) \ A \ A(n, m[n], k[n], \&[n])$
 $(n = 1, 2, 3, \dots)$.
For convenience, set $U_n = U(n, m[n], k[n], \&[n])$. By (2) and Lemma 8,
 U and U_n have some point z in common. For each n , we can choose $n \ n+1 \ n$
an arc y_n in U_{n+1} with one endpoint at z_n and the other at z_{n+1} . Then $y_n \subset Q(n+1,$
 $m[n+1])$. Also,
 $b \in A(n+1, m[n+1], k[n+1], \&[n+1]) \ S=U_{n+1} \ Q(n+1, m[n+1]),$
2 and therefore each point of y_n has distance less than $\frac{1}{2^n}$ from b .
 $\frac{1}{2^n} \rightarrow 0$ as $n \rightarrow \infty$; hence, if we set $y = \bigcup y_n$ then y is an arc $n=1$
with one endpoint at b .
Since U and U_n have a point in common, $n \ n+1 \ r$,
-11 -11
 $\& \ (StJ? \ Pk[n])$ and $\& \ (S \langle JST- \ Pk[n+1] \rangle$
have a common point, and hence ,
 $SC3P \ Pk[n]p$ and $SC3irT' \ Pk[n+i:P \ z \ z$
have a common point. Therefore, if p is the metric on Y , then
 $\frac{1}{2^n} Pk[n]p \leq Pk[n+1]p \leq \frac{1}{2^{n+1}} Pk[n+1]p \leq \frac{1}{2^{n+2}} Pk[n+2]p \leq \dots$
and therefore
 $\lim_{n \rightarrow \infty} \frac{1}{2^n} Pk[n]p = 0$
 $\frac{1}{2^n} Pk[n]p \leq Pk[n+1]p \leq \frac{1}{2^{n+1}} Pk[n+1]p \leq \frac{1}{2^{n+2}} Pk[n+2]p \leq \dots$
Thus $\{Pk[n]p\}$ is a Cauchy sequence and must converge to some point $p \in Y$. Since
 $v_n \text{SVIS}^{f,1}(S(\hat{r} \ Pk\{nt\})) \rightsquigarrow d$
 $Pk[n] ; p-$
 $\lim f(z) = p$. It is possible that y is not a simple arc, but Z^*b
 $z \in Y$ according to [12] we can replace y by a simple arc
 $y' \subset y$. Thus $b \in C$, and we have shown that $B \subset C$.
Suppose $c \in C$. Let y_Q be an arc at c such that f approaches a limit p' along y_Q .
Take any n . Choose k with $p' \in P_k$ • Choose m so that c is in the interior of $Q[h, m)$
 $A \ X$. Then y_Q has a subarc y_Q' , with one endpoint at c , such that
 $Y_o' = \{c\} \ S \ Q(n, m) \ n \ f^{n1}CS(\sim, p_kB)$.

Hence, for some \mathcal{L} , $c \mathcal{L} \text{ acc}[U(n, m, k, \mathcal{L})] = A(n, m, k, s.)$. This shows that

CO CO 00 co

csQUUUA(n, m, k, \mathcal{L}) $n=1^* m \mathcal{L} k=1 \ll 1$

It is easy to deduce from Lemma 7 that the set

$A(n, m, k, \mathcal{L}) - I(U(n, m, k, \mathcal{L})) = \dots$

$A(n, m, k, \mathcal{L}) - [I(U(n, m, k, \mathcal{L})) \cap A(n, m, k, \mathcal{L})]$

contains at most two points. It follows by a routine argument that

$A(n, m, k, \mathcal{L}) - k-J [I(U(n, m, k, \mathcal{L})) \cap A(n, m, k, \mathcal{L})] \cap m, k, \mathcal{L} \cap m, k, \mathcal{L}$

is countable. Since

$[I(U(n, m, k, A)) \cap A(n, m, k, \mathcal{L})] = B \cap m, k, \mathcal{L}$

$\cap m, k, \mathcal{L}$

$C - B$ is countable, and therefore C is of type F_g . **B**

Next we will show that the foregoing theorem is as strong as possible, in this sense: if A is any set of type F_g contained in X , then there exists a bounded continuous complex-valued function f defined in H such that A is the set of curvilinear convergence of f . The proof is unfortunately quite long. .

Definition. Let E^* and E_2 be two sets on the real line. A point p on the real line will be called a *splitting point* for E^* and E_2 if either

$\exists x \in p$ for all $x \in E^*$ and $p \in E_2$ or $\exists x \in E_2$ and $p \in E^*$ for all $x \in E_2$ and $p \in E^*$.

We will say that two sets E^* and E_2 *split*, or that E^* *splits with* E_2 , if and only if there exists a splitting point for E^* and E_2 .

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Lemma 9. Let E be an set in R . Then there is a sequence $\{E_n\}_{n=1}^\infty$ of sets such that

(i) E is bounded and closed

(ii) if $n < m$, then either E_n and E^* are disjoint or E_n and

E_m split

(iii)

Proof. We can

E , and .

CO write $E = \bigcup_{n=1}^\infty A_n$ where $n=1$

A_n is closed, $A_n \subset A_{n+1}$ for all n

Observe that if I is any open interval, then there exists a

CO

countable family $\{J_n\}_{n=1}^\infty$ of bounded closed intervals such that $\bigcup_{n=1}^\infty J_n = I$

$J_n \subset J_{n+1}$ and J_n split, and $I = \bigcup_{n=1}^\infty J_n$. Since any open set of

\mathbb{R}

real numbers is a countable disjoint union of open intervals, it

CO

follows that for any open U there exists a countable family $\{H_n\}_{n=1}^\infty$

of bounded closed intervals

00

such that $n \in I$ and $T \cap$

I split, and $m \in \mathbb{R}$,

$n=1$

, $r \in \mathbb{H}$ 00

For each n , let $\{I_j^n : j \in \mathbb{Z}^+\}$ and

be a family of bounded 00

I_j^n split, and $A = \bigcup_{j=1}^\infty I_j^n$

closed inter

im. Let J

$I^n = \{A_p \cup \{I_j^n : n = 1, 2, \dots; j = 1, 2, \dots\}\}$.

Then $\{I^n\}$ is a countable family of bounded closed sets, and $\mathbb{C} \cap \mathbb{O} \cap \mathbb{O}$

$E \cap \mathbb{A} \cap \mathbb{W} \cup (A_{n+1} \cap A_A)$

$n=1$ $n=1$

$= i^n) = A_{I \cup C}\{$

$n=1$ $] = 1$ $J \cap J$

$\mathbb{K} \cap \mathbb{K}$ -

/ J^*

Let F_1 and F_2 be any two distinct members of \mathcal{F} . If either F_1 or F_2 is $A^c = \emptyset$, then F_1 and F_2 are automatically disjoint. If neither F_1 nor F_2 is A^c , then we can write

$\dots, T \cap (1) \dots, T \cap (2) \dots$

$F_1 \cap X_j(1) \cap A_n(1)+1$, and $F_2 = I_j(2) \cap A_n(2)+1$,

If $n(1) < n(2)$, then $n(1) + 1 \in n(2)$, so

$p_2 = I_j(2) \cap A_n(2)+1 \subset A_n(2) \subset A_n(1)+1$, and therefore F_1 and F_2 are disjoint. A similar argument shows that if $n(2) < n(1)$, then F_1 and F_2 are disjoint. Thus, if F_1 and F_2 are not disjoint, then $n(1) = n(2)$ and we have

$F_1 = I_j(1) \cap A_n(1)+1 \subset I_j(2) \cap A_n(2)+1$,

where $n = n(1) = n(2)$. But then $j(1) \neq j(2)$, so $I_j(1) \cap I_j(2) = \emptyset$, and $I_j(1) \cap I_j(2) = \emptyset$

split, and therefore F_1 and F_2 split. So we have shown that any two distinct members of either split or are disjoint.

If p has infinitely many distinct members, let E_1, E_2, E_3, \dots be a counting of p . If p has only finitely many distinct members, let E_1, \dots, E_k be the members of p and let $E_{k+1} = \emptyset$ for $k > m$. In either case, $\{E_n\}$ is the desired sequence. \square

If F is a closed subset of the real line, then by a *complementary interval* of F we mean a component of F^c . (If $F = \mathbb{R}$, then \emptyset is considered to be a complementary interval of F .)

Definition. By a *special family* we mean a family of subsets of \mathbb{R} such that

(3) is nonempty

(4) each member of \mathcal{F} is bounded and closed

(5) there exists a sequence $\{F_n\}$ of members of \mathcal{F} such that every member of \mathcal{F} is equal to some F_n , and the following condition is satisfied.

(5a) If $m > n$, then either F_m is contained in one of the complementary intervals of F , or else F splits with F .

Lemma 10. If E is an F^\wedge set in R , then there exists a special family jP such that $E = LJ$.

• 00

Proof. By Lemma 9 we can choose a sequence $\{E_n\}_{n=1}^\infty$ of bounded closed sets such that if $n \in I_m$ then E and E_n either split or are disjoint, ¹ $n \in m$ ⁴⁷

00

and $E = \bigcup_{n=1}^\infty E_n$

Let $n_1 = 1$ and let $F_i = E_{n_i}$. Now suppose that n_1, n_2, \dots, n_{t-1} are chosen and F_1, F_2, \dots, F_{t-1} are chosen so that

(i) $1 = n_1 < n_2 < \dots < n_s$

(ii) F_i is closed and bounded ($i = 1, \dots, s$)

(iii) if $n_s > r > t > 1$, then either F^\wedge is contained in one of the complementary intervals of F , or else F^\wedge splits with F^\wedge

(iv) if $1 \leq i \leq n_s$, then there exists $j \leq s$ such

We construct $F \setminus F$ as follows. Let \mathcal{F} be the family of

n_s+1 n_s+1

complementary intervals of the bounded closed set

We assert that E_j meets at most finitely many members of \mathcal{F} . If this

assertion is false, then there exists an infinite sequence $\{I_n\}_{n=1}^\infty$ of members of \mathcal{F} such that $n \in m$ implies $I_n \subset I = \bigcup_{n=1}^\infty I_n$, and there exists

(for each m) a point $x \in I \cap E$ such that $\{x_m\}_{m=1}^\infty$ is a bounded sequence, ⁴⁷ $m \in s+1$ $m \in i$

and $n \in i$ implies that $x_m \rightarrow x$. From this it follows that $\{x_m\}_{m=1}^\infty$ has either a strictly increasing or a strictly decreasing convergent

CO subsequence. We will assume that it is a strictly increasing

convergent subsequence; the reasoning is similar

in the case of a

strictly decreasing convergent subsequence. Say $x_m \rightarrow x$.

$x_m(k) \rightarrow x$ we let

$x_m(k) \rightarrow x$ $x_m(k)$

$x_m(k+1)$

$x_m(k) < x$, so since $x_m(k) < x$, $x_m(k+1)$

$x_m(k+1)$

we must have $x_m(k) < x_m(k+1)$

and

Therefore, if

lim

$x_m(k)$

then x

lim $x_m(k)$

Moreover, for $k \geq 2$

finite real
number

so

that $a_k \in \mathbb{R} \setminus \mathbb{Q}$. Therefore there exists $u \in \{1, \dots, s\}$
such that $a_k \in E$ for infinitely many values of k . Consequently KU .

$x \in E$. But since $x \in E$, $x \in E$ so that E and E

u

E and $E \cup u$ that are less

n

must split $s+1$

Since infinitely

than x ;

and E

, $x \in E$. also. But then $x \in E \cap E \cup S+1 \cup S+1$

and x must be a splitting point for

many

also

a_k lie in E , E_u contains points less than x ;

therefore E and E .

cannot split

and we have a contradiction. This

$u \leq s+1$

proves the assertion.

$\mathcal{J} = \{(J) \cup \{I_n : 1 \leq n \leq s+1\}\}$.

Let n_{s+1} equal n_g plus the number of members of \mathcal{J} . Let F_{n+1}, \dots, F be all the members of \mathcal{J} . We must show that conditions (i) through (v) are still satisfied when s is replaced by $s+1$.

Conditions (i) and (ii) are obvious. The verification of (iii) is divided into three parts. Suppose $n_{g+1} > r > t > 1$.

Case I. Assume that $n_{g+1} > r > t > 1$. In this case we already know that either F_r is contained in one of the complementary intervals of F_t or else F_r splits with F_t .

Case II.

Assume that n

$n > r > n_{s+1}$

$t > 1$.

There exists $v \in \{1$

, $s\}$

such that $F_t \in E_v$. Either E_v and E_{g+} are disjoint or they split.

Case IIa. Assume E^+ and E_{s+j} are disjoint. Either $F^* = \langle j \rangle$ (in which case F^+ is certainly contained in a complementary interval of F) or else $F \not\subseteq I$ and $F = I \cap E$ for some $I \in \mathcal{I}$. Let J be the smallest $r-1 \leq r \leq s+1$

such that $J \cap I^s \neq \emptyset$,

closed interval containing F . Then $J \not\subseteq I$ and $J \not\subseteq I \cap C(E)$, so

$T \cap I \neq \emptyset$

$r-1 \leq r$

that J does not meet E . The endpoints of J lie in $F \subseteq E$, so $r-1 \leq r \leq s+1$,

neither endpoint of J lies in E_v . So J does not meet E_v and therefore

J does not meet F ; from which it follows that F^+ is contained in a complementary interval of F^+ .

Case IIb. Assume that E and E^+ split. Since $F \in E$ and $F \not\subseteq E \cap v_{s+1} \cap t \sim v_{r \leq s+1}$ it follows that F and F^+ split, $t \leq r$

Case III. Assume that $n \leq r \leq t \leq n$. If either F or F^* is $d \geq s+1 - s \leq t \leq T$, it is clear that F_r is contained in a complementary interval of F^+ .

Otherwise, there exist I, I^+ such that $I \cap I^+ = \langle 1 \rangle$ and

$$F_r = Z_1 \cap E_{s+1} \quad F_t = 2^{nE_{s+r}}$$

Since I^+ and I evidently split, F^+ and F^+ must split.

Thus condition (iii) is verified. /

As for (v), it is clear that $s \leq V^{s+1}$

$$E \cap U \cap E \cap C_2 \cap F \cap C \cap q, s+1 \leq i \leq n+1 \leq s+1$$

$$1 \leq 1 \leq s$$

so that

Hence

Thus we

• 'co

have shown that we can construct sequences $\{n_k\}, \dots, p$

$< v$

co $k=1$

in such

a way that conditions (i) through (v) are satisfied

for every value of s . If we set $\hat{F} = \{F_k : k = 1, 2, \dots\}$, it is easy

to verify that \hat{F} is a special

family and that $E = \bigcap \hat{F}$

Definition.

If \hat{F} and \hat{F}^+ are two families of sets, let

$$\hat{F} \cap \hat{F}^+ = \{F_i\}$$

$$F_2 : F_1 \cap \hat{F} \cap F_2 \cap \hat{F}^+$$

Lemma 11. If \hat{F} and \hat{F}^+ are two special families, then $\hat{F} \cap \hat{F}^+$ is a special family.

Proof. Conditions (3) and (4) in the definition of a special family are clearly satisfied, so we just have to verify (5).

Arrange all pairs of positive integers in a sequence according to the scheme shown in Figure 1. Let $(a(k), b(k))$ be the k th term of the sequence'''' ($k = 1, 2, \dots$). Observe that $k < l$ if and

^The reader may find it amusing'to derive the following formulas for $(a(k), b(k))$. For real t , let τ denote the largest integer that is *strictly* less than t . Then

$$\begin{aligned} a(k) &= \left\lceil \frac{1}{2} \left(\sqrt{8k+1} + (-1)^{\lfloor \frac{1}{2} \sqrt{8k+1} \rfloor} \right) \right\rceil - k + 1 \\ &= \left\lceil \frac{1}{2} \left(\sqrt{8k+1} + (-1)^{\lfloor \frac{1}{2} \sqrt{8k+1} \rfloor} \right) \right\rceil - k + 1 \\ &= \left\lceil \frac{1}{2} \left(\sqrt{8k+1} + (-1)^{\lfloor \frac{1}{2} \sqrt{8k+1} \rfloor} \right) \right\rceil - k + 1 \text{ if } \sqrt{8k+1} \text{ is odd} \\ &= \left\lceil \frac{1}{2} \left(\sqrt{8k+1} + (-1)^{\lfloor \frac{1}{2} \sqrt{8k+1} \rfloor} \right) \right\rceil - k + 1 \text{ if } \sqrt{8k+1} \text{ is even, and} \end{aligned}$$

(1,2)
(1,3)
a $\frac{1}{2} \sqrt{8k+1}$
(2,1)
(2,2)
(2,3)
(2,4) $\frac{1}{2} \sqrt{8k+1}$ • •
 Q, l
(3,3)
(3,4) • • •
L4.1)
(+.2)
(4,3)
(4,4) • • •

Figure 1.

only if either $a(k) + b(k) < a(l) + b(l)$ or else $a(k) + b(k) = a(l) + b(l)$ and $b(k) < b(l)$. Thus $k < l$ implies that either $a(k) < a(l)$ or $b(k) < b(l)$.

Let $b^e a$ sequence of elements of f such that every

member of \hat{f} is equal to some F_n and such that condition (5a) in the

• "• $\frac{1}{2} \sqrt{8k+1}$ 00

definition of a special family is satisfied. Let $\{F_n\}_{n \in \mathbb{N}}$ be a similar sequence for Set $F_k = F_a(k) \cup F_b(k)$,

Then $f \setminus K$, is a sequence in such that every member of $J \cap K$ is

is equal to some F , . We must show that condition (5a) is satisfied.

Suppose that $l > k$. Two cases occur.

Case I. $a(k) < a(l)$.

Note that $F^{\wedge} \wedge F_a(k)$ and $F \mathcal{L} \text{---} F_a(\mathcal{L})$, Either $F_a(\mathcal{L})$ is contained in one of the complementary intervals of F . (in which case F . is contained in a complementary interval of F^{\wedge}), or else F^{\wedge} and $F_a(fc)$ split (in which case F . and F . split).

Case II. $b(k) < b(\mathcal{L})$.

In this case a similar argument shows that either F_0 is contained in

$$\begin{aligned} b(k) &= |(2!@pJi \text{---} /SF* *. \text{---}^2) + k \\ &= 4(\gg<tF|] + I - i(-1) ^{[[,/}^^{1]}) ([y^T - | - \\ &|(-1)U^{/§E*T^{\wedge})} + k \\ &4(/8FT + 1)(/8k?T - 1) + k \\ &- 2) + k \\ &\text{if } /8k+1 \text{ is odd} \\ &\text{if } [[Vsk+1]] \text{ is even.} \\ &I \end{aligned}$$

a complementary interval of F^{\wedge} or $F\mathcal{L}$ and F^{\wedge} split. Thus condition (5a) is satisfied, and $J^{\wedge}A^{\wedge}$ is a special family. \square

Lemma 12. Let E_1, E_9 be two F , sets in R such that $E_1 \subset E_9$, and suppose that and are special families such that $E^{\wedge} = \text{ali}d^{\wedge}2 =$ Then $E^{\wedge} = \bullet$

The proof is obvious.

Next we introduce some notation.

Let J be a nonempty interval on X with endpoints a, b ($a \mathcal{L} b$).

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By $\text{Trap}(J, e, 0)$ (where $0 \in (0, y)$ and $e > 0$) we mean the interior z of the trapezoid shown in Figure 2. That is,

$$\text{Trap}(J, e, 0) = \{ (x, y) : 0 < y < e, a + y \text{ctn } 0 < x < b - y \text{ctn } 0 \}.$$

7T

For $0 \leq (0, \wedge)$, let $\text{Tri}(J, 0)$ be the closed triangular area shown in Figure 3. That is,

$$\text{Tri}(J, 0) = \{ \langle x, y \rangle : y \geq 0 \text{ and } a + y \text{ctn } 0 \mathcal{L} x \mathcal{L} b - y \text{ctn } 0 \}.$$

7T

If $x \in X$, $e > 0$, and $0 \in (0, y)$, let $S(x, e, 0)$ denote the open $0 \leq z \leq$

Stolz angle shown in Figure 4. That is,

$$S(x_0, e, 0) = \{ \langle x, y \rangle : 0 < y < e, X_q + y \text{ctn } (it - 0) < x < x_q + y \text{ctn } 0 \}.$$

If K is a closed set on a real line, let $J(K)$ be the smallest closed interval containing K . If K is bounded, closed, and nonempty, $e > 0$, and $0 < B < a < p$ then we define

$$B(K, e, a, B) = \text{Trap}(J(K), e, a) - \cup \text{Tri}(I, B),$$

where \cup denotes the set of complementary intervals of K .

x axis

Figure 2. — $\text{Trap}(J, E, 0)$

x axis

X_{axis}

x axis

Figure 4.— $S(x_0, \ell, 9)$

$I >$ We state without proof the following readily verifiable facts.

(6) $B(K, e, a, 3)$ is an open subset of H .

(7) $S(e, e, 0)$ is an open subset of H .

(8) If K_1 and K_2 split, then for any $e_2, a, 3$, $B(K_1, e_2, a, 3)$ and $B(K_2, e_2, a, 3)$ are disjoint. /

(9) Suppose that $\ell(K, e) > e^* > 0$, and $0 < 3 < 3^* < a < y$.

Then

$B(K_1, E_p) \cap (X_p \cap H \cap B(K, e, a, 3)) = \emptyset$.

(10) Suppose K_j is contained in one of the complementary intervals of K , and suppose $e, a, 3$ are given. Then there exists $\delta > 0$ such that for every $n \in \mathbb{N}$,

$B(K, e, a, 3)$ and $B(K_n, a, 3)$ are disjoint.

IT L *

(11) Suppose that $a < 0 < x_q \in J(K)$. Then, for any e, E_p

$B(K, e, a, 3)$ and $S(x_q, E_p, 0)$ are disjoint. \square **7f**

(12, Suppose that $x \in K \cap J(K)$ and $-3 < a < 0 < \ell$. Let e be given.

Then there exists $\delta > 0$ such that for every $h \in \mathbb{N}$,

$S(x_q, n/\delta) \cap H \cap B(K, e, a, 3) = \emptyset$.

(13) Suppose that $e < e'$ and $0' < 0$. Then

$S(x_q, s', 0') \cap S(x_q, s, 0) = \emptyset$.

(14) Suppose $x_q \in K$ and $e, a, 3, 9$ are given.. Then there exists $\delta > 0$ such that for every $h \in \mathbb{N}$,

$S(x_q, n, 0)$ and $B(K, e, a, B)$ are disjoint.

(15) If $x_q \in x^*$ and $e, 0$ are given, then there exists $\delta > 0$ such that for every $n \in \mathbb{N}$, $S(x_q, e, \delta)$ and $S(X_p, n, 0)$ are disjoint.

(16) $B(K, e, a, B) \cap X \subset K$.

(17) $S(x_0, e, \delta) \cap X = \{x_0\}$.

2

Definition. If \hat{r} is a special family, let T be the set of all members of \mathcal{F} that have two or more points.

Definition. Let \hat{J} be a special family, let E be the set of all endpoints of intervals $J(F)$ where $F \in \mathcal{F}$, and suppose that $0 < B < a < 0 < H$. By a *pair of special $a, 3, 0$ functions* for we mean a pair (e, δ) , where e and δ are positive real-valued functions, 2 the domain of e is E , the domain of δ is \mathcal{F} , and

(18) for each $n > 0$, there exist at most finitely many $F \in \mathcal{F}$ such that $\delta(F) \geq n$;

(19) for each $n > 0$, there exist at most finitely many $e \in E$ such that $e(e) \geq n$;

(20) if $e, e^* \in E$ and $e < e'$, then

$S(e, c(e), 0)$ and $S(e', E(e'), 0)$

are disjoint;

(21) if $F, K \in \mathcal{F}^2$ and $F \leq K$, then

$B(F, \phi(F), a, B)$ and $B(K, \phi(K), a, B)$ are disjoint;

(22) if $e \in E$ and $F \in \mathcal{F}^2$, then

$S(e, \phi(e), 0)$ and $B(F, \phi(F), a, B)$ are disjoint.

Lemma 13. Let \mathcal{F} be a special family and suppose that $0 < a < 1$. Then there exists a pair of special a, B, ϕ functions for

Proof. Let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of members of \mathcal{F}^2 of the type referred to in condition (5) in the definition of a special family. Let $J^*(n) = \{F \in J(\mathbb{R})^2 : F = F^{\wedge} \text{ for some } k \leq n\}$

$E =$ set of all endpoints of intervals $J(F)$ for $F \in J^*(n)$

$E(n) = \{e \in E : e \text{ is an endpoint of } J(F^{\wedge}) \text{ for some } k \leq n \text{ for which } F^{\wedge} \leq F\}$

If $J(F_1)$ has one endpoint e , set $\phi(e) = 1$. If $J(F^{\wedge})$ has two endpoints e_1, e_2 , then by (15) we can choose $e_1^* \leq 1$ and $e_2^* \leq 1$ so that $S(e_1, e_1^*, 0)$ and $S(e_2, e_2^*, 0)$ are disjoint. If $F_1 \in \mathcal{F}^2$, set $\phi(F_1) = 1$. In this case, $J(F_n)$ has two endpoints e and e_0 and (by (11)) $B(F_1, \phi(F_1), a, B)$, $S(e, \phi(e), 0)$ and $S(e_0, \phi(e_0), 0)$ are all disjoint.

Now suppose that $\phi(e)$ and $\phi(F)$ have been defined for all

n

$e \in E(n)$ and all $F \in J^*(n)$ in such a way that

(i) if $e, e' \in E(n)$ and $e \leq e'$, then $S(e, \phi(e), 0)$ and

$S(e', \phi(e'), 0)$ are disjoint;

(ii) if $F, K \in J^*(n)$ and $F \leq K$, then $B(F, \phi(F), a, B)$ and $B(K, \phi(K), a, B)$ are disjoint;

n

(iii) if $e \in E(n)$ and $F \in J^*(n)$ then $S(e, \phi(e), 0)$ and

$B(F, \phi(F), a, B)$ are disjoint;

(iv) if $e \in E(n)$ and $k \leq n$ is the least integer for which

$e \in E(k)$, then $\phi(e) \leq \phi(F)$

n

(v) if $F \in J^*(n)$ and $k \leq n$ is the least integer for which $F \in J^*(k)$, then $\phi(F) \leq \phi(F)$

We must extend the definitions of ϕ and ϕ to $E(n+1)$ and

n

$J^*(n+1)$ in such a way that conditions (i) through (v) are still satisfied when n is replaced by $n+1$.

n

Case I. If $F = *$ or if $F = F$ for some $k \leq n$, then $J^*(n+1) = J^*(n) \cup \{F\}$ and $E(n+1) = E(n)$, so that nothing is required to be done.

Case II. If F consists of a single point e and if $e \leq F$ for some $n+1 \leq k$

$k \leq n$, then (since F and F must split in this case) e is an $n+1$ endpoint of $J(F)$, so that again $J^*(n+1) = J^*(n) \cup \{F\}$ and $E(n+1) = E(n)$, K

and nothing is required to be done.

Case III. Suppose that $J(F)$ consists of a single point e_Q and that

for each $k < n$, $e \in F_k$. By (14), (15), and the fact that $E(n)$ and $J(n)$ are finite, we can choose $e(e_0) \in \mathcal{L}(0, \cdot)$ so that $S(e_Q, e(e_Q), 0)$ is disjoint from $S(e, e(e), 0)$ and from $B(F, 6(F), a, B)$ for each $e \in E(n)$ and each $F \in \mathcal{F}^2(n)$. The construction is then finished for

$E(n+1)$ and $F(n+1)$.

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Case IV. Suppose that F_{n+1} contains at least two points and that, for each $k < n$, F_k splits with F_{n+1} , or else F_{n+1} is contained in a complementary interval of F_k . Since $J(n)$ is finite, (8) and (10) show that we can choose $e_0 \in \mathcal{L}(0, \cdot)$ so that $B(F_{n+1}, e_0, a, 6)$ is disjoint from $B(F, 6(F), a, B)$ for each $F \in \mathcal{F}^2(n)$.

Say $e \in E(n)$. Then e is an endpoint of $J(F_k)$ for some $k < n$, so (since F_{n+1} either splits with F_k or is contained in a complementary interval of F_k) $e \in J(F_{n+1})^*$. By (11), $B(F_{n+1}, e_0, a, 6)$ and $S(e, e(e), 0)$ are disjoint.

Let e_1, e_2 be the endpoints of $J(F_n)$.

$e_1 < e_2$

Case IVa. $e_Q \in \mathcal{L}(n)$.

In this case the construction is already finished.

Case IVb. $e_Q \in E(n)$ and $e_Q \in \mathcal{L}(n)$.

If $e \in \mathcal{L}(F_k)$ for some $k < n$, then F_k splits with F_{n+1} , so that $e \in J(F_{n+1})^*$.

e_Q must be an endpoint of $J(F_{n+1})$ —which contradicts the assumption that $e \in E(n)$. Hence, for each $k < n$, $e \in \mathcal{L}(F_k)$. By (14), (15), and the fact that $E(n)$ and $J(n)$ are finite, we can choose

$e_0 \in \mathcal{L}(0, \cdot)$

so that $S(e_0, e(e_0), 0)$ is disjoint from

$S(e, e(e), 0)$ and from $B(F, 6(F), a, B)$ for each $e \in E(n)$ and each $F \in \mathcal{F}^2(n)$. By (11), $S(e_Q, e(e_Q), 0)$ and $B(F_{n+1}, 6(F_{n+1}), a, B)$ are disjoint. Thus the construction is finished for $E(n+1)$ and $F(n+1)$.

/

Case IVc. $e_Q \in E(n)$ and $e_Q \in \mathcal{L}(n)$.

This case is essentially the same as Case IVb.

Case IVd. $e_Q \in E(n)$ and $e_Q \in \mathcal{L}(n)$.

$e_Q \in \mathcal{L}(n)$.

$k < n$, then F_k splits with F_{n+1} , so

If $e \in F_k$, for some $k < n$

e_Q is an endpoint of $J(F_k)$; a contradiction. Thus $e \in F_k$ for $k < n$, and similarly $e \in F_k$ for $k < n$. Therefore, by (14) and (15), we can choose $e(e_0)$ and $e(e_1) \in \mathcal{L}(0, \cdot)$ so that $S(e, e(e), 0)$ and $S(e_0, e(e_0), 0)$

$S(e_0, e(e_0), 6)$ are disjoint and each of $S(e_Q, e(e_Q), 9)$ and

$S(e_0, e(e_0), 9)$ is disjoint from every $S(e, e(e), 9)$ ($e \in E(n)$) and from every $B(F, 6(F), a, 8)$ ($F \in \mathcal{F}^2(n)$). By (11), $S(e_Q, e(e_Q), 9)$ and

$S(e_0, e(e_0), 9)$ are each disjoint from $B(F_{n+1}, a, 8)$ so that

construction is finished for $E(n+1)$ and $(r(n+1))$.

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We have shown that we can inductively define $e(e)$ for every $e \in E$ and $\phi(F)$ for every $F \in \mathcal{F}^2$ in such a way that (i) through (v) are satisfied for every value of n . Conditions (20), (21) and (22) in the definition of a pair of a, B , 9 special functions are thus automatically satisfied by (e, ϕ) . We must verify that (18) and (19) are also satisfied.

Suppose (19) is false. Then there exists $n > 0$ and there

$\{e_k\}$ exists an infinite sequence distinct members of E such that

$e_k \in J(F)$ for every k . Let $m(k)$ be the least integer for which e_k is an endpoint of $J(F, \dots)$. Each $J(F)$ has at most two endpoints, so, $m(k)$ in \mathbb{N}

since the e_k are all distinct, there exists (for given m) at most two values of k for which $m(k) = m$. Therefore there exist infinitely

many distinct integers among $m(1), m(2), m(3), \dots$. Consequently

there exists j with $m(j)$

$< n$. But, by (iv), $e(e_j) \in J(F)$

contradiction. So (19) must be true. A similar argument shows that

(18) is true. \square

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Lemma 14. Let T be a special family, $0 < g < a < 0 < y$, and. let E be the set of all endpoints of intervals $J(F)$ for $F \in \mathcal{F}^*$. Suppose (e, ϕ) is a pair of special $a, g, 0$ functions for If $e \in E$, are two real-

valued functions having domains E and \mathbb{R} respectively, and if

$0 < E(e) \leq e(e)$ for all $e \in E$, and

$0 < \phi(F)$ for all $F \in \mathcal{F}^2$,

then $(\text{Sp } \phi)$ is a pair of special $a, 0, 0$ functions for \mathbb{R} .

Proof. This follows from the fact that

$S(X_o, E', 0) \leq S(X_o, E'', 0)$ I

and $B(K, e', a, g) \leq B(K, e'', a, g)$

\mathbb{R} whenever $e' \leq e''$. \square

Theorem 4. Let A be any set of type F_{ag} in X . Then there exists a . bounded continuous complex-valued function f defined in H such that A

is the set of curvilinear convergence of f .

co

Proof. We can write $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is of type F and $|A_n| \leq n$

$n=1$

$A_{p+1} \cap A_n$ for every n . For each n , let \mathcal{F}_n be a special family with CX_n -valued

-31

$\mathcal{F}_n \cap \mathcal{F}_{n+1}$ for $n \in \mathbb{N}$,

By Lemmas 11 and 12, together with mathematical induction, K is a special family and \mathcal{F}_n is a subset of some member of \mathcal{F} .

\square

Let $\{x_n\}$ be a strictly ascending sequence in $(0, \infty)$

IT

converging to g .

Let $\{x_n\}$ be a strictly descending sequence in (∞, ∞) \bullet^{11} converging to ∞ $\circ 7T$, ,

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Let $\{x_n\}$ be a strictly ascending sequence in \mathbb{Q} , $\bullet \text{---}$

converging to g .

Let E be the set of all endpoints of intervals $J(F)$ for

Let $(e(1,0), 5(1,-))$ be any pair of special $o^\wedge, 8^\wedge, 6_1$ functions for .

Now; suppose that for each $k < n$ we have chosen a pair of

special $a, 3, 6$, functions $(e(k,-), 6(k,-))$ for ft . in such a way $K K K i^*$

that

' (i) whenever $1 \leq k \leq n-1$, $eg^{E_{k+p}} F g$

$e \in F \cap J(F)$, then

$S(e, e(k+1, e), 6_{k+1}) \text{ AHCIW } 6(k, F), <x_k, 3_k);$

(ii) whenever $1 \leq k \leq n^* - 1$, eg

., and $e g E.$, then $k+1^* k'$

$SCe, e(k+1, e), 6_{k+1}) \text{ nHCs}(e, e(k, e), 6_k);$

(iii) whenever $1 \leq k \leq n-1$, $Kg-J^{\wedge}_{+1}, Fe^{\wedge 2}, K \text{---} F \gg$ then

$B(K, 6(k+1, K), a_{k+1}, B_{k+1}) \text{ AHCB}(F, 6(k, F), a_R, 3_k).$

Then we construct $(e(n+1,-), 6(n+1,-))$ as follows. Let

$(e, 6)$ be any pair of special $a_{n+p} \wedge_{n+1}^{9_{n+1}}$:Funct^{ions} $\wedge^\circ \wedge_{n+i}^{i'} \wedge^{-2}$ *

$e g E_{n+i} - E_n$, then for some unique $F \in , e \leq F \in J(F)$, so by (12) we can choose

5(e) > 0 such that $n \leq 5(e)$ implies

$S(e, n, 0_{n+1}) \text{ nHGB}(F, 6(n, F), a_n, \&_n).$

We set $e(n+1, e) = \min \{c(e), 5(e)\}$. On the other hand, if $e g E_{n+} \wedge r \mid E^\wedge$, then we set $e(n+1, e) = \min \{e(e), e(n, e)\}$.

If then there exists a unique K with $F \leq K$.

Set

$6(n+1, F) = \min \{6(F), \sim 6(n, -K)\}.$

By Lemma 14, $(e(n+1,*)) > < 5(n+1,')$ is a pair of special $a p$

$B., 6$, functions for \wedge ., and by (13) and (9), conditions (i), /

$n+1 \ n+i \ n+i$ (ii), and (iii) are still satisfied when n is replaced by $n+1$. Thus we can inductively construct a pair $(e(n,)> 6(n,))$ of special $a_n, \&_n, 0_n$ functions for in such a way that conditions (i), (ii) and

(111) are satisfied for every n .

Let

$V_{-J}^{SO} > e^{(n \gg e)} > {}^0 J \mid V e \in E., J$

$VL^{BCF} > 6^{<n> F})^{\omega} \in t \ n \ n$

Then U_n is open. For fixed n , all the various sets $S(e, e(n, e), 0^\wedge)$ ($e \in E$) and $B(F, 6(n, F), a_n, \&_n)$ ($F \in 5^\wedge$) are open and pairwise disjoint, so that every component of U_n is contained in one of the sets $\wedge, 2$

$S(e, e(n, e), 0_n)$ ($e \in E_n$) or $B(F, 6(n, F), \langle n, \&_n \rangle)$ (Fifty. It therefore follows from (16) and (17) that if Q is any component of U , then

$$(23) \ll AX A_n.$$

From the fact that $(e(n, '), 6(n, *))$ is a pair of special $a_n, B_n, \textcircled{C}_n$ functions for 5^\wedge together with conditions (18) and (19), it follows that

$$u_n^{nH} = tU S(e, e(n, e), 6) HH] u e 6 E n \dots\dots\dots$$

$$[U_2 B(F, 6(n, F), a_n, y n H].$$

Consequently, conditions $Ci) > (ii), (iii)$, together with the fact that $e \in E, - E e 6 F n J(F) \text{ for some } ^1$

$$n+1 n^{1J} n'$$

show that $U, H U$ for every $n. n+1 n$

By Urysohn's Lemma, there exists a continuous function

$g_n : H \boxplus [0, 1]$ such that

$g_n(z) = 1$ for $z \in H - U_n$ and $g(z) = 0$ for $z \in TT' _ n H.$

$$\boxplus n n+1$$

co

Let $g(z) = S - s(z)$. Then $0 < g(z) \leq 1$, and the series converges $n=1 2^n$ uniformly, so g is continuous on H .

If

$z \in H - U_n$, then $z \mathcal{L} H -$ for every $m > _ n$

so

that

and hence

$$(24)$$

CO

$$. g(z) 1^E m=n$$

$$1 = 1$$

$$2^m 2^{n-1}$$

$$(z \in H - U_n) .$$

Also

if $z \in U, n+1'$

then $z \in U, U \gg, \dots, U$, so that $12 n+1$

$= g_n(z)$, and

$$(25)$$

$$. i^z ns m=n+1 2^u$$

$$1$$

$$2^n$$

$$\in U_{n+1}$$

We assert that

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(26) for each $x_q \in A$, $g(z) \rightarrow 0$ as $z \rightarrow x_q$ with $z \in S(x_q, \delta)$ •

Take any natural number n . Since $x \in A$, $\exists \delta > 0$ either $\delta > 1/n$ or $\delta \leq 1/n$ case, set $n = 1/\delta$. In the second case, (12) shows that we can choose $n > 0$ so that

$S(x_q, \delta) \cap S(x_0, \delta) \neq \emptyset$.

Suppose $x \in S(x_q, \delta)$ and $y \in S(x_0, \delta)$. Then, in the first case, $x, y \in S(x_0, \delta)$ and $S(x_q, \delta) \cap S(x_0, \delta) \neq \emptyset$, and in the second case,

$x, y \in S(x_0, \delta)$ and $S(x_q, \delta) \cap S(x_0, \delta) \neq \emptyset$. So, by (25), $(x, y) \in S(x_0, \delta)$ and $y \in S(x_0, \delta)$ implies $x \in S(x_0, \delta)$.

$0 \leq g(x, y) \leq \delta$

This proves (26). ¹

Let x_q be a point in X and y any arc at x_q . Suppose $g(z) \not\rightarrow 0$

as $z \rightarrow x_q$ along y . Then y has a subarc y' with one endpoint at x_q

such that $y' \cap \{x_q\} = \{x_q\}$. Therefore, by (23), $x_q \in A$.

$1 \leq \delta$.

$2^n, 2^n$ Since

0). By (24), $y' \cap \{x_q\} = \{x_q\}$.

n was arbitrary, $x_q \in A$.

$1 \leq n \leq \infty$

$= A$.

Thus,

(27) if there exists an arc y at x_q such that $g(z) \not\rightarrow 0$ as $z \rightarrow x_q$ along y , then $x_q \in A$.

Now define

$f(x, y) = \frac{1}{2} g(x, y) \sin \theta + i g(x, y) \cos \theta$ ($(x, y) \in H$).

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If $x_q \in A$, then, by (26), $f(z) \rightarrow 0$ as $z \rightarrow x_q$ with $z \in S(x_q, \delta)$.

Thus every point of A is in the set of curvilinear convergence of f .

Conversely, suppose x_q is any point of the set of curvilinear

convergence of f . Let y be an arc at x_q such that f approaches the limit $c + di$ along y . Then g approaches the limit d along y . If d is different from zero, then $g(x, y) \sin \theta$ (the real part of f) cannot approach any limit along y — a contradiction. Therefore g approaches the limit 0 along y , and, by (27), $x_q \in A$. Therefore A is the set of curvilinear convergence of f . \square

5. Boundary Functions for Continuous Functions

Lemma 15. Let E be a metric space, Y a separable metric space. i

Suppose that $f: E \rightarrow Y$ is a function having the following property. For every open set $U \subseteq Y$ there exists an F_σ set $L \subseteq E$ and a countable set N such that

$f^{-1}(U) \cap L \subseteq \overline{f^{-1}(U) \cap N}$.

Then there exists a countable set $M \subseteq E$ such that $f|_M$ is of class e -M

$CF_a(E - M)$.

Proof. Let B be a countable base for Y . For each $B \in \mathcal{B}$, let $L(B)$ be an F^σ set and let $N(B) \subseteq E$ be a countable set such that $\langle^1(B) \subseteq L(B) \subseteq \langle^{p^{-1}}(B) \cup N(B)$. Let $M = \bigcup_{B \in \mathcal{B}} N(B)$. Then M is countable. Let $E = E - M$ and let $B \in \mathcal{B}$ if $\langle^1(B) \subseteq E$. We show that $\langle^1(B)$ is of class $(F(E_q)) \cdot \sigma$. Let W be any open subset of Y . If $p \in W$, there exists $r > 0$ such that $S(r, p) \subseteq W$. Choose $B \in \mathcal{B}$ so that $p \in B \subseteq S(r, p)$. Then $B \subseteq W$. It follows that $N = \bigcup B = \bigcup_{B \in \mathcal{B}} B \subseteq d(W)$ where $(J(W) = \{B \in \mathcal{B} : B \cap W \neq \emptyset\})$. Therefore $V_b^{-1}(W) = E \setminus \bigcup_{B \in \mathcal{B}} B = E \setminus M = E \setminus \bigcup_{B \in \mathcal{B}} N(B) = E \setminus M$. Consequently $\langle^1(W) = E \setminus M = E \setminus \bigcup_{B \in \mathcal{B}} N(B)$, so $\langle^1(W)$ is of class $W^{(1)}$. Theorem 5. Let Y be a separable metric space and let $f : H \rightarrow Y$ be a continuous function. Suppose that $E \subseteq X$ and that $\langle^1 : E \rightarrow Y$ is a boundary function for f . Then there exists a countable set $M \subseteq E$ such that $\langle^1 \cdot \cdot$ is of class $(F(E - M))$. Proof. Let U be any open subset of Y , and let $W = (U)'$. Set $E_n = \{x \in X : \text{there exists an arc } y \text{ at } x, \text{ having one endpoint on } X_n, \text{ such that } y - \{x\} \cap f^{-1}(U) \neq \emptyset\}$. $K = \{x \in X : \text{there exists an arc } y \text{ at } x \text{ such that } Y - \{x\} \subseteq f^{-1}(W)\}$. Observe that $\bigcap_{n=1}^\infty E_n = K$ and $\langle^1 p^{-1}(W) \subseteq K$. For the time being, let n be a fixed natural number. For each $x \in K$ we can choose an arc y_x at x such that $Y_x - \{x\} \subseteq H_n \cap f^{-1}(W)$. Since an arc at x is by definition a simple arc, $y_x - \{x\}$ is a connected set and hence must be contained within one nonempty component of $H_n \cap f^{-1}(W)$. Let U denote this component (for each $x \in K$). $n \cdot x$ Let T be the set of all points of K that are two-sided limit points of E_n . We claim that if $x, y \in T$, then $x \mid y$ implies $U_x \cap U_y = \emptyset$. If $U_x \cap U_y \neq \emptyset$, then (since U_x and U_y are two components of the same set) U_x and U_y are equal. Let p be the endpoint of y_x lying in U and X .

let

q be the endpoint of y lying in $U^\wedge = U_{x<}$

We can

join p and q by an arc y lying in $U \setminus X$

Putting y , y and y together, $x y$

we obtain an arc a with one endpoint at x and the other at y , such that $a - \{x, y\} \cap U' = \emptyset$. According to [12] we can choose a simple arc

$a' \subset a$ having one endpoint at x and the other at y . Of course, $a' - \{x, y\} \cap U^\wedge = \emptyset$. Let I be the open interval in X with endpoints at x and y , and let $J = X - I$. Let B be the bounded component of $H - a'$ and let A be the other component. Since X_n is unbounded and does not meet a' , $X \cap A = \emptyset$.

Because x is a two-sided limit point of E_n , we can choose a point $w \in I \cap E_n$. Let g be an arc at w , having one endpoint on X_n , such that $g - \{w\} \subset U^\wedge$. Then g does not meet a' (because $a' - \{x, y\} \cap U^\wedge = \emptyset$ and $f \setminus w \cap U^\wedge = \emptyset$), and therefore (since $g - \{w\}$ contains a point of $X \cap A$) $g - \{w\} \subset A$. It follows that

$w \in A$. This, however, is a contradiction

because the frontier of A

(relative to the finite plane) is $a' \cup J$.

We conclude that, for $x, y \in T$, countably many components,

$x y$ implies U^* . $C \setminus U = \langle j \rangle$.

An open set in the plane has only

so it follows that T must be countable. Let S be the set of all points of E_n that are not two-sided limit points of E_n .

We know that

S is countable, so

$K \cap E_n = [K \cap (E - S)] \cup [K \cap S]$

$= T \cup [K \cap S]$

is countable. \square

Let $N = K \cap \bigcup_{n=1}^{\infty} E_n = (K \cap E)$. Then N is countable, and, $\bigcup_{n=1}^{\infty} E_n = E$

since $\langle p \rangle \cap (W) \cap K = \emptyset$.

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$\langle p^{-1}(U) \cap E \rangle \cap J \cap E \subset E$

$\bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty}$

00 CO

$= (E \cap K \cap E) \cup ((E - K) \cap E) \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty}$

$(E \cap N) \cup (E - K) \subset (E \cap N) \cup (E - (p^{-1}(U) \cap E))$

$= (E \cap N) \cup \langle p^{-1}(U) \rangle$.

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Thus $\langle p^{-1}(U) \cap E \rangle \cap \bigcup_{n=1}^{\infty} E_n \subset (E \cap N) \cup \langle p^{-1}(U) \rangle$, and the desired result $\bigcup_{n=1}^{\infty}$ follows from Lemma 15. \square

Corollary. Let Y be either the Riemann sphere, the real line, or a finite-dimensional Euclidean space. If $f : H \rightarrow Y$ is a continuous function, if $E \subset X$, and if $q \in E \rightarrow Y$ is a boundary function for f , then cp is of honorary Baire class 2 (E, Y) .

Next we show that the foregoing corollary is as strong as possible in the sense that if E is any subset of X and $(p$ is a function of honorary Baire class 2 mapping E into a suitable space, then there exists a continuous function in H having f as a boundary function. A proof of this result—at least for real- or vector-valued functions was outlined by Bagemihl and Piranian [2, Theorem 8], in the case

where $E = X$. Although the construction given here is carried out much l

more explicitly than the construction given by Bagemihl and Piranian, my treatment differs from theirs in only two aspects that are of any significance. First of all, the proof of the theorem for arbitrary subsets E of X depends on Lemma 6 of the Introduction. Secondly, Bagemihl and Piranian say in the last line of their proof that there is "no difficulty now in extending f continuously to the whole of D in such a manner that $\langle j \rangle$ is a boundary function for f ." While this appears to be all right for real- or vector-valued functions, it is not clear why the extension should be so easy for functions taking values on the Riemann sphere. Theorem 7 of the present paper shows, however, that the result can be obtained for functions taking values on the sphere once it is known for vector-valued functions.

The following miniature closed graph theorem will be a convenience.

Lemma 16. Suppose that M is a metric space and that $u : M \rightarrow R$ is a function having the following properties:

(i) if $\{p_n\}$ is a convergent sequence of points of M , then $\{u(p_n)\}$ converges neither to $+$ nor to $-$

(ii) if $\{p_n\} \subset M$, $p \in M$, and $y \in R$, and if $p_n \rightarrow p$ and $n \rightarrow \infty$ $u(p_n) \rightarrow y$ then $u(p) = y$.

$n \rightarrow \infty$ Then u is continuous.

Proof. Suppose that $\{p_n\}$ is a sequence of points in M converging to a point $p \in M$. Using (i) it is easy to show that $\{u(p_n)\}$ is a bounded sequence. Suppose that $\{u(p_n)\}$ does not converge to $u(p)$. Then there exists a subsequence $\{u(p_{n_k})\}$ that converges to a real number $y \neq u(p)$. This, however, contradicts (ii). We conclude that

$u(p_n) \rightarrow u(p)$ as $n \rightarrow \infty$.

Lemma 17. Let $h : R \rightarrow R$ be a strictly increasing function such that $h(R)$ is neither bounded above nor bounded below. Then there exists a $1c \neq$

continuous weakly increasing function $h : R \rightarrow R$ such that $h(h(x)) = x$ for every $x \in R$.

Proof. Let $Z = h(R)$. Observe that $h^{-1} : Z \rightarrow R$ is strictly increasing. For any $x \in R$, the set $(-\infty, x] \cap Z$ is nonempty. Also, $h^{-1}((-\infty, x] \cap Z)$ is bounded above, because if we choose $y \in Z$ with $x \leq y$, then

$h^{-1}((-\infty, x] \cap Z)$ is bounded above by $h^{-1}(y)$.

We claim that for every $x \in R$

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$$(27) \sup h((- \ll, x] \cap Z) = \sup h((- \ll, x) \cap Z).$$

If $x \notin Z$, the equation is trivial. Suppose $x \in Z$. Then $y < h^{-1}(x)$ ($h(y) < x$ and $h(y) \in Z$), .

so that $h((- \ll, h^{-1}(x))) \subset (- \ll, x) \cap Z$. Hence

$$(- \ll, h^{-1}(x)) \subset (- \ll, x) \cap Z,$$

so that $\sup h^{-1}((- \ll, x) \cap Z) \geq h^{-1}(x) = \sup h^{-1}((- \ll, x] \cap Z)$. The opposite inequality is trivial, so (27) is established.

We also claim that

$$(28) \inf h^{-1}((x, +\infty) \cap Z) = \sup h^{-1}((- \infty, x] \cap Z).$$

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Obviously, $\inf h((x, +\infty) \cap Z) \geq \sup h((- \infty, x] \cap Z)$. Take any $y > \sup h((- \infty, x] \cap Z)$. If $h(y) \leq x$, then $h(y) \notin (- \infty, x] \cap Z$, and

so $y \notin h((- \infty, x] \cap Z)$ — a contradiction. Thus $h(y) > x$ and $h(y) \in (x, +\infty) \cap Z$. Therefore $y \leq h^{-1}((x, +\infty) \cap Z)$, and so $\inf h^{-1}((x, +\infty) \cap Z) \leq y$. In view of the choice of y , this implies

that

$$\inf h^{-1}((x, +\infty) \cap Z) \leq \sup h^{-1}((- \infty, x] \cap Z), \text{ and (28) is established.}$$

Define

$$h^*(x) = \sup h((- \infty, x] \cap Z).$$

* [§]

It is clear that h is weakly increasing and that $h(h(x)) = x$ for $x \in I$ every real x . The continuity of h can easily be deduced from the equations

"k ft

$$\sup h((- \infty, x)) = h(x) \inf h^*((x, +\infty)) = h^*(x),$$

which are established as follows:

$$\sup h^*((-\infty, x)) = \sup \sup h^{-1}((-\infty, y] \cap Z)$$

$$= \sup h^{-1}((- \ll, x) \cap Z)$$

$$= \sup h^{-1}((- \ll, x] \cap Z)$$

, * □

$$= h(x)$$

$$\inf h((x, +\infty)) = \sup h((- \ll, y] \cap Z)$$

$$= \sup h^{-1}((y, +\infty) \cap Z)$$

$$= \inf h^{-1}((x, +\infty) \cap Z)$$

$$= \sup h^{-1}((- \infty, x] \cap Z),$$

* *

$$= h(x). \square$$

Theorem 6. Let E be any subset of X and let $\phi: E \rightarrow \mathbb{R}$ be any function of Baire class 2 (E, \mathbb{R}). Then there exists a continuous function $f: H \rightarrow \mathbb{R}$ such that ϕ is a boundary function for f .

Proof. Let $\phi: E \rightarrow \mathbb{R}$ be a function of Baire class 1 (E, \mathbb{R}) and N a countable subset of E such that $\phi(x) = \lim_{n \rightarrow \infty} \phi(x_n)$ for every $x \in E - N$. Let $\{s_n\}$ (with $n + m$ implying $s_n < s_m$)

Let S be a countable dense subset of X that includes every integer and every point of \mathbb{N} . Let

$$t = 1 \text{ if } s \text{ is an integer}$$

$$t = 0 \text{ if } s \text{ is not an integer,}$$

$$n \leq 2^n, n \leq 6$$

Define

$$h(x) = \sum_{0 < s_n < x} t \text{ if } x > 0$$

$$i. h(x) = -\sum_{x < s < 0} t \text{ if } x \leq 0.$$

Then h is a strictly increasing function from \mathbb{R} into \mathbb{R} , and $h(\mathbb{R})$ is
 * bounded neither above nor below. Let h be the function described in

Lemma 17.

Suppose that $0 < y < 1$. Then (for fixed x)

$$*, x - (1-y)u, \dots$$

$$u - h\left(\frac{x - (1-y)u}{y}\right) -$$

is a strictly increasing continuous function of u that approaches $+\infty$ as $u \rightarrow +\infty$ and $-\infty$ as $u \rightarrow -\infty$. Consequently there exists precisely one number $u(x, y)$ that satisfies the equation

$$(29) \quad u(x, y) - h\left(\frac{x - (1-y)u}{y}\right) = 0.$$

I claim that $u(x, y)$ is a continuous function on

$$= \{ (x, y) : x, y \in \mathbb{R} \text{ and } 0 < y < 1 \}.$$

Suppose $\{ (x_n, y_n) \} \subset H_1$ and $y_n \rightarrow y^*, y^* \in H_1$. If $u(x_n, y_n) \rightarrow u^*$ then $x_n \rightarrow y^* u^*$.

and hence

$$*, x - (1-y)u(x, y) - h\left(\frac{x - (1-y)u}{y}\right) >$$

$$n, y_n \rightarrow y^* \text{ which contradicts (29). Thus } u(x_n, y_n) \text{ cannot approach } +\infty.$$

A similar argument shows that $u(x_n, y_n)$ cannot approach $-\infty$. Now assume that

$$u(x_n$$

Then, by (29)

$$\lim_{n \rightarrow \infty} u(x_n, y_n) = u^*$$

$$x_n \rightarrow x, y_n \rightarrow y$$

$$x - (1-y)u(x, y)$$

$$*, x - (1-y)u$$

$$- h\left(\frac{x - (1-y)u}{y}\right) = 0$$

so $u \rightarrow u^*$

$$= u(x$$

$$y) \bullet$$

By Lemma 16, u is continuous.

From

Lemma 6, there exists a sequence of continuous functions mapping X into such that $g^n(x) \rightarrow^* ip(x)$ for each $x \in E$. n

For $n > 2$

define

$$f_n(x, y) = (y_n(n+1) - n)g_n(u(x, y)) + ((n+1) - y_n(n+1))g_{n+1}(u(x, y))$$

$$\begin{cases} 1 & \text{when } -\infty < y < -\frac{n+1}{2} \\ 0 & \text{when } y \geq -\frac{n+1}{2} \end{cases}$$

Then f is o

continuous

on

we can assume that

is

r_n

o

$\inf_{x \in S} x$

n

2^{-n}

defined

$h(x)$

By the Tietze extension theorem

and continuous on

all

of H . Let

v_n

$\sup_{x \in S} x$

$h(x)$

$V(s_n) - K_{S_n}$

if $s \in N$ n

v_n

if $s \in N$. $n \in T$

If x

and y are real numbers, define

$$x \vee y = \max\{x$$

$y\}$

set

$$A_n(x, y) =$$

$$[(1 - ny) \vee 0] [(1 - \frac{1}{n}) \cdot X A/$$

$n \in$

$$I \cap A = 2s$$

$$1 \leq n \leq n$$

$$s \rightarrow x$$

$$2^{-2} y$$

Then

A_n

is continuous in H . Observe that $A_n(x, y) = 0$ when $y > \frac{1}{n}$

Using this fact, it is easy to show that, if we set

$$f = f + \sum_{n=1}^{\infty} A_n$$

then f is defined and continuous in H . We now show that f is a boundary function for f .

Let p be any point of ∂H . The line

$$(30) \quad x = (h(p) - p)y + p$$

passes through $(p, 0)$, and the part of it that lies in H is an arc at p . We will show that f approaches $f(p)$ along this line. If we substitute $(h(p) - p)y + p$ for x in the expression for $A_n(x, y)$, we obtain

obtain

$$(31) \quad A_n(x, y) =$$

$$[(1 - ny) \vee 0] [(1 - |h(p) - p|) - 2(1 - |h(p) - p|)(s - p)] y^n$$

If $p \leq s$, then $h(p) \leq s$, and one can verify directly that (31) vanishes. If $p > s$, then $h(p) \leq r$, and again one can verify directly that (31) vanishes. Thus $A_n(x, y)$ vanishes along that part of the line (30) lying in H .

Solving (30) for $h(p)$, we find that, along the given line, $h(p) =$

and hence $p = h(h(p)) = h(h(p))$. Therefore, if $0 < y < 1$, $p = u(x, y)$. So, if \hat{x}, \hat{y} satisfies (30), $n \leq 2$, and $\hat{y} \leq \frac{1}{n}$, then

$$f_0(x, y) = (y^{n+1} - y^n)g_n(p) + ((n+1) - y^{n+1})g_{n+1}(p).$$

Since the coefficients of $g_n(p)$ and $g_{n+1}(p)$ in the above expression add up to 1 and since both coefficients lie in $[0, 1]$, $f(x, y)$ lies on the line segment joining $g_n(p)$ to $g_{n+1}(p)$. It follows that $f_Q(x, y)$ approaches $f(p)$ as \hat{y} approaches p along the line (30). This line lying in H , $f(x, y)$ shows that f approaches $f(p)$ that lies in H . Again, we first consider the value of f along the

Since each A_n vanishes on the part of ∂H also approaches $f(p)$ along the line.

Let s be any point of N . We follow the part of the line r . A

$$(32) \quad x = (s - p)y + p$$

given line. Substituting the value of x given by (32) into the expression for A_n , we obtain

$$(33) \quad A_n(x, y) =$$

$$[(1 - ny) \vee 0] [(1 - |s - p|) - 2(1 - |s - p|)(s - p)] y^n.$$

If $s < p$, then $\ell < r < \ell < r$, and one can verify directly that (33) vanishes. If $s > p$, then $\ell < r < \ell < r$, and again one can verify that (33) vanishes. Thus, for $n \leq m$, $f(x, y) = 0$ when \hat{x}, \hat{y} lies on the line

(32) and in H .

If we take $n = m$ in (33), we obtain

$$A_m(x, y) = [(1 - my) \vee 0] y^m.$$

Therefore $A_m(x, y)$ approaches $v_m = 9(s_m)$ - along the given line. Take any $\langle x, y \rangle \in$ satisfying (32), and take any a and b satisfying

$$(34) \ a < s < b.$$

m

$\hat{x} < r \wedge \mathcal{L} h(b)$, so that

Then $h(a) \cdot \text{ft} \cdot \text{---} m$

$(h(a) - s)y + sm \cdot m$

$x < (h(b) - s)y + s$; from which we deduce that $m \cdot m$

$h(a)$.

$x - (l-y)s$

*

Since h

is weakly increasing, $^{**} x - (lry)s$

$\cdot h(h(a)) < h(\text{---} \text{---})$

m^{**}

$\text{---}) \cdot < h(h(b)) = \cdot b$.

Since a

and b were

taken to

be any two numbers satisfying (34), we

conclude that

whence it follows that $u(x, y) =$

s . Thus m

$f_{\mathcal{O}^{(x)}}$

$y) = (yn(n+1) - l\mathcal{O}g^s) +$

when

$f_{\mathcal{O}^{lx}}$

1 1

$\langle x, y \rangle$ lies on the given line and $\mathcal{L} y \mathcal{L} y$ - Consequently y approaches $ip(s_m)$ along the line (32). So f approaches $\langle p(s_m) +$

$\langle p(s) \cdot \wedge(s_m) = \langle p(s_m) \rangle$ and the theorem is proved. \square

2

Theorem 7. Let E be any subset of X and let $cp: E \rightarrow S$ be any

2

function of honorary Baire class 2(E, S). Then there exists a

2

continuous function $f: H \rightarrow S$ such that $\langle p$ is a boundary function for

f .

Proof. The proof of this theorem is very similar to that of Theorem 1.

2 3

Since $S \subset \mathbb{R}^n$, there exists, by Theorem 6, a continuous function

3

$g : H \rightarrow \mathbb{R}$ having $g|_S$ as a boundary function. Let

$$K = g^{-1}(\{v \in \mathbb{R}^3 : |v| = 1\})$$

$$L = g^{-1}(\{v \in \mathbb{R}^3 : |v| > 1\})$$

$$F = g^{-1}(\{v \in \mathbb{R}^3 : |v| < 1\}).$$

Let $g_0 = g|_S$. H is homeomorphic to \mathbb{R}^n , so by [5, Lemma 2.9, p. 299], g can be extended to a continuous function g_1 on

$$g_1 : H \rightarrow \{v \in \mathbb{R}^3 : |v| = 1\}.$$

3

Define $f^* : H \rightarrow \mathbb{R} \setminus \{0\}$ by setting

$$f^*(z) = g(z) \text{ if } z \in L$$

$$f^*(z) = g_1(z) \text{ if } z \in F.$$

Then, since F and L are closed, f^* is continuous on H . It is easy to

verify that f^* is a boundary function for g . Let $P_Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$

2

be the 0-projection onto S (see page 11), and let f be the composite

2

! function $P_Q \circ f^*$. Then f maps H continuously into S , and $P_Q \circ f^*$ is a boundary function for f . \square

CHAPTER II. BOUNDARY FUNCTIONS FOR DISCONTINUOUS FUNCTIONS

6. Boundary Functions for Baire Functions

It is not known whether the set of curvilinear convergence of a Borel-measurable function defined in H is necessarily a Borel set. The answer is not known even for functions of Baire class 1. However, a theorem on boundary functions that is similar to the corresponding result for continuous functions in H can be proved for functions of Baire class 5 in H .

Definition. If A and B are two sets, we will call A and B *equivalent* and write $A \sim B$ if and only if $A \setminus B$ and $B \setminus A$ are both countable. It is easy to check that \sim is an equivalence relation.

Lemma 18. If $A \sim E$, then $S \setminus A \sim S \setminus E$ for any set S . If $A \sim E$, then for all n in some countable set N , then

$$L \setminus A \sim L \setminus E \text{ and } p|_A \sim p|_E \text{ for } p \in N \text{ and } n \in N.$$

The proof of this lemma is routine.

Definition. An interval of real numbers will be called *nondegenerate* if it contains more than one point.

Lemma 19. Any union of nondegenerate intervals is equivalent to an open set.

Proof. Let \mathcal{I} be any family of nondegenerate intervals. It will suffice to prove that $\bigcup \mathcal{I}$ is countable. We can write

$\mathcal{I} = \bigcup_{n \in \mathbb{N}} \mathcal{I}_n$
 where $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$
 is a
 countable family of disjoint open intervals.

If $x \in \bigcup \mathcal{I}$
 then $x \in I$
 so that
 is
 an endpoint of I
 for some $I \in \mathcal{I}$. For some n , $I \in \mathcal{I}_n$
 \mathcal{I}_n^*
 \mathcal{I}_n
 is an endpoint of \mathcal{I}_n^* .

$\bigcup \mathcal{I}_n^* \subset \bigcup \mathcal{I}$
 Thus
 is
 contained in the
 set of all endpoints of
 the
 various \mathcal{I}_n . and the lemma is
 proved.

Lemma 20. Let h be a weakly increasing real-valued function on a nonempty set E . Suppose that $|x - h(x)| \leq 1$ for every $x \in E$. Then h can be extended to a weakly increasing real-valued function \hat{h} on \mathbb{R} .

Proof. Let $e = \inf E$ (e may be $-\infty$). For each $x \in (e, +\infty)$, set
 $\hat{h}(x) = \sup \{h(t) : t \in E, t < x\}$.
 Since $|t - h(t)| \leq 1$ for each $t \in E$,
 $t \in E, t < x \implies h(t) \leq x - 1$,
 so \hat{h} is finite-valued. If $e = -\infty$ we are done. If $e > -\infty$ then $x \in E$ implies $h(x) \geq x - 1$, so h is bounded below. For $x \in (-\infty, e]$ set
 $\hat{h}(x) = \inf h(E)$.

It is easy to verify that h^\wedge has the required properties. \square

Lemma 21. Let Y be a metric space, $f : R \rightarrow Y$ a function of Baire class 5(R, Y), and suppose that $h : R \rightarrow R$ is weakly increasing. Then there exists a countable set N such that the composite function $f \circ h|_D$ is of Baire class 5($R - N, Y$). \square

Proof. Let N be the set of discontinuities of h . By a well-known theorem, N must be countable. But then $h|_R$ is continuous, so that $f \circ (h|_R) = (f \circ h)|_R$ is of Baire class 5($R - N, Y$). \square

Lemma 22. Let Y be a separable arcwise connected metric space, E any metric space, and let $\langle p : E \rightarrow Y \rangle$ be a function having the following property. For every open set $U \subseteq Y$ there exists a set $T \in P^{++}(E)$ such that $\langle p^{-1}(U) \rangle \subseteq T$. Then, if $\mathcal{L} \geq 2$, $\langle p \rangle$ is of Baire class 5(E, Y).

Proof. The proof is similar to that of Lemma 15. Let \mathcal{B} be a countable base for Y , and suppose that W is any open subset of Y . Let

$$d(W) = \{U \subseteq W : U \in \mathcal{B}\}.$$

The argument in the proof of Lemma 15 shows that $w = u = \bigcup_{U \in d(W)} U$.

For each $U \in d(W)$, let $T(U) \in P^{++}(E)$ be chosen so that $\langle p^{-1}(U) \rangle \subseteq T(U)$. Then $\langle p^{-1}(W) \rangle = \bigcup_{U \in d(W)} T(U)$.

$$d(w) \subseteq \bigcup_{U \in d(W)} T(U)$$

$$d(w) \subseteq \bigcup_{U \in d(W)} T(U)$$

Thus $\langle p^{-1}(w) \rangle = \bigcup_{U \in d(W)} T(U)$, and since $P^{++}(E)$ is closed under countable unions,

$$\langle p^{-1}(w) \rangle \in P^{++}(E)$$

$$\langle p^{-1}(w) \rangle \in P^{++}(E)$$

unions, $\langle p^{-1}(W) \rangle \in P^{++}(E)$. Therefore $\langle p \rangle$ is of Baire class $\mathcal{L}(E, Y)$. \square

Theorem 8. Let Y be a separable arcwise connected metric space, $f : H \rightarrow Y$ a function of Baire class $g(H, Y)$, and $\langle p : E \rightarrow Y \rangle$ a boundary function for $C + 1(E, Y)$.

Proof.

Observe

that

Y is a separable arcwise connected metric space,

Y where

f . Then

Let U be any open subset of

$$C = A \cup B.$$

For each x

and let $V =$

$g \geq 1$, E a subset of

Y is of

$Y - U$.

$= V^{-1}(V)$
 choose an arc y_v
 Baire class
 Set
 at x such
 $\lim_{z \rightarrow x} f(z) \in Y_x$
 Y_x
 Y_x
 $|z - x| \leq 1\}$
 $- \{x\} \subset f^{-1}(V)$
 Notice that if $x \in A$ and
 We will say that
 have subarcs y' and $y' \cdot x' \cdot y$
 if
if
 $x \in B$.
 $y \in B$, then
 y , meets y .
 \mathbf{X}
 in
 respectively such
 $n, 311(1 Y_X, n Y_Y^* *$
 $L = \{x \in A : (\forall n)(\exists y)(y \in C, a$
 $M = \{x \in a$
 $= \{x' \in$
 Let
 and
 $: (\forall n).(\exists y)(y \in C$
 $: (\exists n)(v \text{ meets no } x$
 $: (\exists n)(y \text{ meets no$
 $L \wedge ULh$
 $M = M_a \cup .$
 and
 $y'' * \bullet$
 provided that y and $y \times y$
 that $x \notin y_x' \subset H_n$,
 meets y in H)} $y' \cdot x \cdot n'$
 meets y
 $\text{in}_{H_n})\}$
 $Y_y \text{ (with } y \mid x) \text{ in } H_n)\}$
 $Y_y \text{ (with } y \mid x) \text{ in } HJ$

Observe that $L_\&, 1^\wedge, M_\&$, are pairwise disjoint, and that $B = L^\wedge \cup M_\&$.

For

meets no y

meets no y

each $x \in M$, let $n(x)$ be a positive integer such that y

(with y

x) in \cdot . Then $n \geq n(x)$ implies that y_x

Let

\bullet meets X_n , and, if $x \in M$, $n > n(x)$ }.

Then $K \subset \cdot$ for each n , and $C = \bigcap_{n=1}^\infty X_n \subset M$.

$n=1$

We next show that for each positive integer n and each x there exists a nondegenerate closed interval I_n such that

$x \in C \cap I_n$. By the definition

$y \in C \cap I_n$ such that y meets y_x in

interval having its endpoints at

x and

y -

of L , there exists a'

Let I^n be the closed x

Let t be any point of

We must prove that $t \in L_\& \cup (X - K)$. Then y meets X_{n+1}

that y must meet either y or $y'x'$.

$K \cap I^n$

If $t \in K$ we are done. $\sim n$

and hence it is clear from

rigorized by means of

Theorem 11.8

in (This argument can be found on p. 119 in [11].) But,

then (because $t \in K_n$)

$n \geq n(t)$, so

Therefore

$t \in M$. Now

Hence y

$C \cap B = A$.

$I^n \cap x$

So

Figure 5

be

if $t \in M$,

that this situation is impossible.

$x \in L \cap A$, so, since y intersects y , a \bullet $y'x'y^*$

Similarly, since y intersects y or y
 $\boxtimes x \cdot y$

tec
 $= A.$

Thus $t \in A - M = L$, and we have shown that a

$(X -$
 $K).$

Let W_n

$x \in L$

For each n ,

a

$L \subset W_n \subset Q[L \cup (X - K)] \subset A$, $C \cap L \cap d \cap$

and therefore

$Vac_{n=1} \text{ co}$

$\boxtimes \{p \mid [L \cup (X - K_n)]\} \subset c$,

$n=1$ co

$- I_a^{L \cup (X - C[K_n])} \subset C$

CO.

$= (L \cap C) \cup (C - U K) = L \cup \langle j \rangle = L \dots$

$d \subset t \cap CL \cap d$

$n=1$

co

It follows that $L = (fA \ W_n) \cap C$. By Lemina 19, each is equivalent $n=1$.

to an open set, so there exists a (\mathbb{S} set $G_a \subset Q \times X$ such that

I

$L \cap G \subset c \cap a \cap a$

$\bullet \bullet \boxtimes$

A similar argument shows that there exists a (\mathbb{S} set $G^\wedge \cap X$ such that

$L_b \subset \mathcal{O}_{AC_-}$

Next we study the properties of M . In doing this, it is 81 ‘

2 convenient to define a function $tt : R \boxtimes^* R$ by setting $ir(x, y) = x$.

If $x \in M \cap K$, then, starting at x and proceeding along y , let $p(x) \in \bullet \bullet x \cap I^*$

be the first point of X that is reached. Define $h^\circ : M \cap K \rightarrow \bullet^* R \cap n \cap n$

by setting $h^\circ(x) = ir(p_n(x))$. If $x, x' \in M \cap a$ and $x < x'$, then, since

y cannot meet y' in H , it is evident that $p(x)$ must lie to the left of $p(x')$; that is, $ir(p_n(x)) < Tr(p_n(x'))$. Thus h° is a strictly

increasing function on $M \cap K^\wedge$. Moreover, \bullet

$|x - h^\circ(x)| \leq 1$ because $y_x \subset \{z : |z - x| \leq 1\}$.

So, by Lemma 20, h° can be extended to a weakly increasing function $h : X \rightarrow R$. Let n

$g_n(x) = f(h_n(x), 1)$ ($x \in R$).

$f(x, \cdot)$ is a function (of x) of Baire class $\mathcal{L}(X, Y)$, so, by Lemma 21, there exists a countable set $N \subseteq X$ such that $g|_N$ is of Baire class n .

Then g_n is of Baire class n .

$n=1$

$\in (M \setminus N, Y)$.

For $x \in M \setminus K_n$, $g_n(x) = f(h^\circ(x), 1) = f(p_n(x), 1)$. If $x \in M$, then for all sufficiently large n , $x \in M \setminus K_n$, so

$\lim_{n \rightarrow \infty} f(p_n(x), 1) = f(x, 1)$.

(pl., hence $\langle p_n, x \rangle$ is of Baire class n in $M \setminus N$).

Thus $g|_M$ is of Baire class $n+1$.

Obviously

L

so $\lim_{n \rightarrow \infty} g_n(x) = f(x, 1)$.

N, Y). It follows that there exists $D \subseteq P^{n+2}(X)$ such that $a(m-n) = (v|_{M \setminus N})^{-1}(u)$

$= DA(M \setminus N)$.

$A \subseteq M \setminus DA(M)$. Now,

$= L \cup L \cap fG \subseteq A \cap U \cap fG$. $A \cap U = fG \cup G$, $1 \subseteq A \cap C$

$a \cap 'a' \cap 'd' \cap 'a' \cap 'd'$

so

I

$M_a = A \cap M \setminus P \cap M = D \cap A \cap (C \setminus L)$

$= D \cap A \cap [C \setminus ((G_a \cup G_j) \cap A \cap C)]$

$= D \cap A \cap [x \setminus (G_a \cup G_j) \cap a \cap c]$

G_a and G_j are G , so $X \setminus (G_a \cup G_j)$ is \emptyset , and hence

$x \setminus (G_a \cup G_j) \in P^2(X) \cap P^{n+2}(X)$.

Therefore $M_a \subseteq F \cap A \cap C$, where $F \in P^{n+2}(X)$. Now, $G_a \in G_g(X) = Q^2(X)$, and since $E, > 1$, $Q^2(X) \subseteq P^{5+2}(X)$, so $G \cup F \subseteq P^{C+2}(X)$. But a

$A = L \cup M \setminus (G \cap A \cap C) \cup (F \cap A \cap C) = (G \cup F) \cap A \cap C$, $A \subseteq A \cap C$

so $A \setminus S \subseteq \emptyset$, where $S \in P^{n+2}(X)$. Since every countable set is F , it is now easy to show that

$A = T \cap C$

for some $T \in P^{n+2}(X)$. From the definition of C it follows that

$T \subseteq X \setminus B$. Thus we have

$= A \cap T \cap C \setminus B = E \cap C \setminus B =$

$T \cap G \cap P^{n+2}(E)$, so Lemma 22 shows that g is of Baire class $n+1$ in (E, Y) . \square

Corollary. Let Y be a separable arcwise-connected metric space, $f: H \rightarrow Y$ a Borel-measurable function, E a subset of X , and $\langle p: E \rightarrow Y$ a boundary function for f . Then vp is Borel-measurable.

Proof, f is of some Baire class $5CH, Y$, hence $\langle p$ is of Baire class $g + 1(E, Y)$, hence $\langle p$ is Borel-measurable. \square

This corollary raises the question of whether a boundary function for a Lebesgue-measurable function is necessarily Lebesgue-measurable, which we answer in the next section.

7. Boundary Functions for Lebesgue-Measurable Functions

i

Suppose that $a_Q, b, a^{\wedge}, b^{\wedge}$ are extended real numbers, and that $a < b, a; < b..$ To make the formalism more convenient we let $0 \rightarrow 0^* 1 \rightarrow 1$

$(-\langle \rangle) - (-\rangle) = 0$ and $(+\langle \rangle) - (+00) = 0$. In other respects we adhere to the usual conventions regarding arithmetic operations that involve $-co$ or $+00$. Let

$$T(a_Q, b_Q, a_1, b_1) = \langle : 0 \leq y \leq 1 \text{ and}$$

$$(a_1 - a_Q)y + a_Q \leq x \leq (b_1 - b_Q)y + b_Q \rangle.$$

A set of this form will be called a *closed trapezoid*. We also consider $\langle | \rangle$ to be a closed trapezoid. A set S will be called a *trapezoid* if there exists a closed trapezoid T such that $T^1 \subset S \subset T$, where T^1 denotes the interior of T relative to H^{\wedge} . Every trapezoid is Lebesgue-measurable, though not necessarily Borel-measurable. *

If s, s' are disjoint line segments having endpoints $a_Q, 0^{\wedge} \langle a_1, 1 \rangle$, and $\langle a_Q', 0 \rangle, \langle a_1', 1 \rangle$ respectively, where $a^{\wedge} \leq a^{\wedge'}$ ($i = 0, 1$), then let

$$T(s, s') = T(s', s) = T(a_Q, a_Q'; a_1, a_1').$$

If $s = s'$, then we let $T(s, s') = T(s', s) = s$. In what follows we will use the symbol X_Q as an alternative designation for the x -axis X . This will enable us to make statements about $X^{\wedge} (1=0, 1)$ (where X_j denotes, as before, $\{ \langle x, 0 : x \in R \}$).

We omit the proofs of the following two routine lemmas.

Lemma 23. Let the line segments s', s_-, s^{\sim}, s . each have one endpoint $X \subset O$ on X_Q and the other on X^{\wedge} , and assume that $i \neq j$ implies that either $s_i \cap s_j = \emptyset$ or $s_i = s_j$. If $T(s_i, s_j) \cap T(s_k, s_l) \neq \emptyset$ then

$$T(s_i, s_j) \cap T(s_k, s_l) \neq \emptyset$$

$$T(s_1, s_3) \subset T^{\wedge}, s_2) \cup T(s_3, s_4).$$

Lemma 24. Let be any set of line segments, each of which has one endpoint on X_Q and the other on X^{\wedge} , and no two of which intersect.

Then $T(s, s')$ is a trapezoid.

$$s, s' \in \mathcal{L}_2$$

Let m denote two-dimensional Lebesgue measure in R . If E

$$2 \subset Z$$

is a measurable subset of some line in R , let $m(E)$ denote the linear

$$\text{---} \mathcal{L}$$

Lebesgue measure of E . Let m_g and m_l denote two-dimensional exterior measure and linear exterior measure, respectively; i.e., for any $E \subset \mathbb{R}^2$;
 $m(E) = \inf \{m(U) : E \subset U \text{ and } U \text{ is open}\}$; 6

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and if E is a subset of a line L , then $p' \in p \cup L$

$m^*(E) = \inf \{m(U) : E \subset L \text{ and } U \text{ is open relative to } L\}$.

Theorem 9. Let \mathcal{C} be any set of line segments, each of which has one endpoint on X_q and the other on X_p and no two of which intersect. Let $S = \bigcup \mathcal{C}$. Then

$i \cdot m(S) = \sum (m(U_i \cap X) + m(H_i \cap X))$. $\mathcal{C} \subset U \cup H \subset X$

Proof." We may assume that \mathcal{C} is nonempty. Let ϵ be any positive real number. Choose an open set $U \subset \mathbb{R}^2$ such that $S \subset U$ and

$m(U) \leq m(S) + \epsilon$.

Let $E_i = S \cap X^i$ ($i = 0, 1$). Choose sets $U_i \subset X^i$ that are open relative to X^i such that $E_i \subset U_i$. G and H are

$p \in p$

$m(G_i) \leq m(U_i) + \epsilon$ ($i = 0, 1$).

1 " $\mathcal{C} \subset X$

2

Let V be the union of all lines $L \subset \mathbb{R}^2$ such that L meets both G and H

$G \cap V$ It is easy to show that V is an open set. Furthermore, $S \subset V$ and $V \cap X^i = G_i$ ($i = 0, 1$). Now let $W = U \cap V$. Then $i \cdot m(W) = \sum m(W \cap X^i)$.

W is open, $S \subset W \subset U$, and

$E_i \subset W \cap X^i \subset G_i$ ($i = 0, 1$).

If $s, s' \in \mathcal{C}$, define $s \sim s'$ if and only if $T(s, s') \subset W$.

It is easy to verify by means of Lemma 23 that \sim is an equivalence relation. Let T be the set of all equivalence classes. We prove that T is countable.

If $s \in T$, we let $a(s)$, $b(s)$ be the endpoints of s on X^i

($i = 0, 1$). Then,

2

$s = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1 \text{ and } x = (a(s) - b(s))y + b(s)\}$.

Since s is compact and contained in W , there is no difficulty in showing that there exists $V > 0$ such that

$\mathcal{L} \{x, y \in \mathbb{R} : 0 \leq y \leq 1 \text{ and}$

$(a(s) - b(s))y + b(s) \leq x \leq (a_1(s) - b_1(s))y + b_1(s) + \delta\}$ $X \cup U \subset X \cup V \subset O$

$\mathcal{C} \subset W$.

Let $K(s) = (a(s) - b(s), a(s) + b(s))$ ($i = 0, 1$). A sketch will rapidly convince the reader that if $s, s' \in \mathcal{C}$, $J_Q(s) \cap J_0(s')$ and $J_x(s) \cap J^1(s')$ are disjoint, then $T(s, s') \subset W$, so that $s \sim s'$. Thus

$(J_0(s) \times J_1(s')) \cap (J_0(s') \times J_x(s')) = \emptyset$. $s' \sim s$.

For each $C \in F$, choose $s(C) \in C$ and let

$$Q(C) = J_0(s(C)) \times J^{\text{sf}}(C).$$

Then $C \cap C_9 = \emptyset$. Since each $Q(C)$ is a nonempty open subset of \mathbb{R} , this implies that F is countable.

If $C \in F$, let

$$T(C) = [s(C), s'(C)]$$

By Lemma 24, $T(C)$ is a trapezoid. Also,

$$(35) \quad C \cap T(C) = \emptyset.$$

Suppose that $C \cap C_2 \neq \emptyset$ and we claim that

$$T(C) \cap T(C_2) = \emptyset. \text{ Assume that } T(C) \cap T(C_2) \neq \emptyset. \text{ Then there exist}$$

$$s_n, s'_n \in C \text{ and } s_2, s'_2 \in C_2 \text{ such that } T(s_n, s'_n) \cap T(s_2, s'_2) \neq \emptyset.$$

By Lemma 22,

$$T(s_1, s_2) \subset T(s_1, s'_2) \cup T(s_2, s'_1) \subset W,$$

so that $s_1 = s_2$; a contradiction. Therefore $T(C) \cap T(C_2) = \emptyset$. \square

Let $I(C) = T(C) \cap X_{\pm}$ ($i = 0, 1$). Then $I(C)$ is an interval and

$$(36) \quad E(C) \subset K(C) \subset W \cap X. \quad (i = 0, 1).$$

$$\sup_{x \in I(C)} |f(x)| < \epsilon$$

Furthermore, $C \cap C_0$ implies that $K(C) \cap K(C_0) = \emptyset$. Using the formula for the area of a trapezoid, we find that

$$|I(C) \cap I(C_0)| \leq |I(C)| + |I(C_0)|$$

$$= |C \cap C_0|$$

$$= |s(C) - s'(C)| \cdot |K(C) \cap K(C_0)| \leq |C \cap C_0|$$

$$= |E(C) \cap E(C_0)| = |I(C) \cap I(C_0)|.$$

$$\text{Let } a = |I(C) \cap I(C_0)| \leq |C \cap C_0|$$

$$= |I(C) \cap I(C_0)|.$$

cgr

According to (35), $S \cap T(C) \cap W \cap U$, so that

$$(37) \quad m(S) < a < m(U). \quad \epsilon \in (0, \epsilon_0)$$

By (36),

$$(38) \quad |m^*(E_0) + m^*(E_1)| < a < |m^*(G_0) + m^*(G_1)|$$

$$< 7 \text{CmhE} + m^*(E) + \epsilon.$$

Since ϵ is arbitrary, inequalities (37) and (38) imply that

$$m(S) = |m^*(E) + m^*(E)|. \quad \square$$

One wonders to what extent a result resembling the foregoing theorem might be obtainable without the hypothesis that no two of the line segments intersect. The following example is relevant to this question. Let M_q be a residual set of measure zero in X_q and let E be a residual set of measure zero in X . Let (x_q, y) be any point of H_j . We claim that there is a line segment passing through (x_q, y) that has one endpoint in E and the other in M_q . For $0 \in (0, 1)$, let

$$f_0(t) = (1 - y_0) \cos t + x_0, \quad \text{and}$$

$$f_1(t) = (x_0 - y_0) \sin t, \quad 0 \leq t \leq 2\pi.$$

Then F^\wedge is a homeomorphism of $(0, it)$ onto X^\wedge , so $F_q^{-1}(M_o)$ and $F^\wedge(M^\wedge)$ are both, residual sets in $(0, it)$. Choose a $\mathcal{L}^{F_o^\wedge M_o} \cap F^\wedge(M^\wedge)$. Let L be the line whose equation is

$$x = x_q + (y - y_o) \cot \alpha.$$

Then L passes through the points (x_q, y_q) , $F_Q(a)$ and $F_1(a)$, so that LaH^\wedge is the desired line segment. Let S be the set of all line segments having one endpoint in M_q and the other in M^\wedge . Then $S \cap X_q$ and $S \cap X^\wedge$ both have measure zero, but, as we have just

1 - '

shown, S , so that S has infinite measure. See Problem 5 at the end of this paper.

Lemma 25.

For every $\epsilon > 0$ there exists a strictly increasing real-valued function h on R such that $h(R)$ has

measure zero, and, for every

real x , $|x - h(x)| \leq \epsilon$.

$I \rightarrow I$

Proof. For each integer n , let $I = [ne, (n+1)e]$. Then $I_n = R \setminus \bigcup_{n=1}^\infty I_n$

There exists a strictly increasing function $f : [0, 1] \rightarrow [0, 1]$ such that $m(f([0, 1])) =$

0. For example, such a function may be defined as follows. Any number in $[0, 1)$ may be written in *ternary* form

$0.a_1a_2a_3\dots$ (binary decimal),

A *binary* decimal

where the decimal does not end in an infinite unbroken string of 1's.

Set.

$f(0.a_1a_2a_3\dots) = 0.b_1b_2b_3\dots$ (ternary decimal), where $b_i = 0$ if $a_i = 0$ and $b_i = 2$ if $a_i = 1$. $i = 1, 2, 3, \dots$

Set $f(1) = 1$. Then f maps $[0, 1]$ into the Cantor ternary set, so

$m(f([0, 1])) = 0$. It is easily shown that f is strictly increasing.

For each n , it is easy to obtain from f a function $f_n : I \rightarrow I$ such that

$f_n(I_n) \subset I_n$ and $m(f_n(I_n)) = 0$. Set

$h(x) = f_n(x)$ for $x \in I_n$. There is no difficulty in proving that h has the required properties. \square

Theorem 10. There exists an indexed family $\{Y_\alpha\}_{\alpha \in A}$ of simple arcs $X \subset C(A)$

such that

(i) for each $x \in X$, y is an arc at $x \in X$

(ii) $x \in Y_\alpha \iff y_\alpha \in Y_\alpha$

(iii) $\bigcup_{\alpha \in A} Y_\alpha$ is a set of measure zero, $x \in X$

Proof. For each natural number n , let $h_n : R \rightarrow R$ be a strictly increasing function such that $h_n(R)$ has measure zero and, for every x , $|x - h_n(x)| \leq 1/n$. For every $x \in R$, let $s_n(x)$ be the line segment joining the point $(x, h_n(x))$ to the point $(x, h_{n+1}(x))$. Since

$$\langle h_{n+1}^{tx} 2 \rangle < \sup x < \sup 1 < \langle \sup X < \sup 2 \wedge h_{n+1}^{(X)} 1 \rangle < h_{n+1}^{tx} 2 \rangle,$$

we see that x^\wedge implies $s_n(x^\wedge) \cap S_n^\wedge = \langle | \rangle$. Let $S_n = VJts_n(x) : x \in R\}$. Then

$$S \cap X \subset \{(x, : x \in h(R))\}$$

$$n \cap \dots, n, n^k,$$

$$\text{and } S \cap X \cap C \cdot \{ /x, -A^{-5} : x \in h_n(R) \}, n \cap +1 - x \gg n+1 / n+1^{k \cdot J} *$$

$$g \mathcal{L}$$

so $m'(S \cap X) = m(S \cap X) = 0$. It is easy to deduce from $n \cap n \cap n+1$

Theorem 9 that ;

1'1 If 2

$$V^{S_n} \bullet < K - nJT' T CV^{S_n} \cap V * V^{S_n} \cap X_{n+1} \gg', \bullet$$

For $x \in X$, let $y = \{x\} \cup I) s(x)$. Since $\setminus h(x), \dots / \bullet^* x, y$ is $x \cap n \cap n / n x$

an arc at x .

00

$$(\mathbb{R})_e(U \cap Y_x) < m_e W + m_e(U \cap S J$$

$$^e x \in X^{x \in e} n=1^n \cdot co$$

$$\mathbb{E} m(X) + \mathbb{E} m(S) = 0, ^e n=1 ^e n$$

so y is a set of measure zero. $\boxtimes x \in X^x$

Corollary. Let $\langle p$ be an arbitrary function mapping X into any topologi

cal space Y having an element called 0. Then there exists a function

$f : H \rightarrow Y$ such that $f(z) = 0$ almost everywhere and ipis a boundary

function for f .

Proof. If $\{y\} \cap \gamma \text{th}(\mathbb{R})$ family of arcs described in Theorem 10, let $X \cap X \cap f(z) = 0$ if z is in no y

X

$$f(z) = \langle f \rangle(x) \text{ if } z \in Y \cap A$$

Then f is the desired function. (\mathbb{R})

Corollary. There exists a real-valued Lebesgue-measurable function f defined in H having a nonmeasurable boundary function defined on X .

SOME UNSOLVED PROBLEMS

1. If A is an arbitrary set of type F_{a5} in X , does there necessarily exist a *real-valued* continuous function f defined in H having A as its set of curvilinear convergence? If $\langle p$ is an arbitrary real-valued function of honorary Baire class 2 on A does there exist a continuous real-valued function f defined in H having A as its set of curvilinear convergence and $\langle p$ as a boundary function?

2. (First proposed by J. E. McMillan [10]). If A is any set of type 2

F_{a5} in X and if $\langle p$ is any function of honorary Baire class 2 (A, S) , 2

does there necessarily exist a continuous function $f : H \rightarrow S$ having A as its set of curvilinear convergence and $\langle p$ as a boundary function?

3. If f is a real-valued Borel-measurable function defined in H , is the set of curvilinear convergence of f necessarily a Borel set? What if f is assumed to be of Baire class 1?

• 3 -

4. Let $S = \{ \langle x, y, z \rangle \in R : Z > 0 \}$. If f is a function defined in S , we define the set of curvilinear convergence of f in the obvious way. If f is continuous, is its set of curvilinear convergence necessarily a Borel set? Is it necessarily of type F ?

5. Let \mathcal{L} be a set of line segments each having one endpoint on X_q and the other on X^\wedge , and let $S \subseteq \mathcal{L}$. Assume that S is a Borel set.

$\mathcal{L} \setminus S$

If $m(S \cap X_q)$ and $m(S \cap X^\wedge)$ are known, what lower bound can be given for $m(S)$? A solution to this problem might be helpful in attacking a problem of Bagemihl, Piranian, and Young [3, Problem!].

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**Ted's Work as an Assistant
Professor of Mathematics at the
Uni. of California**

7. 1968 - Note on a Problem of Alan Sutcliffe

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MR0228409 Kaczynski, T. J. Note on a problem of Alan Sutcliffe. Math. Mag. 41 1968 84.86. (Reviewer: B. M. Stewart) 10.05

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NOTE ON A PROBLEM OF ALAN SUTCLIFFE

T. J. KACZYNSKI, The University of Michigan

If n is an integer greater than 1 and $ah, \bullet \bullet, a_b, ao$ are nonnegative integers, let $(ah, \bullet \bullet \bullet, ai, ao)_n$ denote $ahn^h + \bullet \bullet \bullet + ain + ao$.

Thus if $O_{i-1} \dots O_1 i$ ($i = O, \bullet \bullet \bullet, h$), then a_h, \hat{a}, a_1, ao are the digits of the number $(ah, \bullet \bullet \bullet, a_b, ao)_n$ relative to the radix n . Alan Sutcliffe studied the problem of finding numbers that are multiplied by an integer when their digits are reversed (*Integers that are multiplied when their digits are reversed*, this Magazine, ¹

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Similarly,

(3) $6^* = -1$.

Taking $q = \text{characteristic of } F$ ($q=0$), choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

$$(Z + ra + i)(r^2 + 1 + ria + lb) = r(Z^2 + r^* + 1)a + (Z^2 + r^2 + 1)6 = 0.$$

One of the factors on the left must be 0, so for some numbers $u, v, w, u \not\equiv 0 \pmod{q}$, we have $w+va-ub = Q$, or $b = -u^{-1}va - u^{-1}w$. So b commutes with a , a contradiction. We conclude that S is not a generalized quaternion group, so 5 is cyclic.

Thus every Sylow subgroup of F^* is cyclic, and F^* is solvable (¹, pp. 181—182). Let Z be the center of F^* and $\text{accnme } F^*$. Then F^*/Z is solvable, and its Sylow subgroups are cyclic. Let A/Z ("fit.. ZC^{\wedge}) be a minimal normal subgroup of F^*/Z . A/Z is an elementary abelian group of order p^k (p prime), so since the Sylow subgroups of F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be any element of F^* , y any element of A . Since A is normal, $xyx^{-1} \in A$, and $(1+x)y = z(1+x)$ for some $z \in Z$. An easy manipulation shows that $y - z = zx - xy = (z - yx^{-1})x$.

¹ P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*,

If $y - z = z - xyx^{-1} \neq 0$, then $y = z = xyx^{-1}$ so x and y commute. Otherwise, $x(z - xyx^{-1})^{-1}(y - z)$. But A is abelian, and $z, y, xyx^{-1} \in A$, so x commutes with y . Thus we have proven that A is contained in the center of F^* . a contradiction.

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$y \in A(n+1, m[n+1], k[n+1], f[n-1]) \subset U_{n+1} \subset Q(n-1, m[n+1]),$
 $2t \geq 1$

and therefore each point of y_n has distance less than from y . Now
 $\bigcup_{n=1}^{\infty} F_0 \subset S^n$; hence, if we set $y = \{y\} \cup \bigcup_{n=1}^{\infty} y_n$, then y is an arc with
one endpoint at y .

Since U_n and U_{n+1} have a point in common,

$f^{-1}(S^{n-1} \cap P_k[n])$ and $f^{-1}(S^n \cap P_k[n+1])$

have a common point, and hence

$S^{n-1} \cap P_k[n]$ and $S^n \cap P_k[n+1]$

have a common point. Therefore, if p is the metric on K , then

$\frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{2}, \quad P_k[n] \cap P_k[n+1] \neq \emptyset$

8. March 1969 - Boundary Functions for Bounded Harmonic Functions

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MR0236393 Kaczynski, T. J. Boundary functions for bounded harmonic functions. *Trans. Amer. Math. Soc.* 137 1969 203.209. (Reviewer: J. E. McMillan) 30.62 (31.00)

Explanation by John D. Bullough

A function $p(e)$ defined on the unit circle is a boundary function for a function $f(z)$ defined in the unit disk provided for each e , $f(z)$ has the limit $p(e)$ at e along some curve lying in the unit disk and having one endpoint at e . Any two boundary functions for the same function f differ at only countably many points by the ambiguous-point theorem of Bagemihl; and a boundary function for a continuous function differs from some function in the first Baire class at only countably many points. In answer to a question of Bagemihl and Piranian, the author constructs a bounded harmonic function having a boundary function that is not in the first Baire class. He shows that nevertheless the set of points of discontinuity of such a boundary function is a set of the first Baire category.

Article by Ted

BOUNDARY FUNCTIONS AND SETS OF CURVILINEAR CONVERGENCE
FOR CONTINUOUS FUNCTIONS

BY

T. J. KACZYNSKI

Let D be the open unit disk in the complex plane, and let C be its boundary, the unit circle. If $x \in C$, then by an *arc at x* we mean a simple arc γ with one end point at

x such that $y \in \{x\}^\wedge D$. If f is a function mapping D into some metric space M , then the set of curvilinear convergence of f is defined to be

$\{x \in C: \text{there exists an arc } y \text{ at } x \text{ and there exists a point } p \in M \text{ such that } f(z) \rightarrow p \text{ as } z \rightarrow x \text{ along } y\}$.

If f is a function whose domain is a subset E of the set of curvilinear convergence of f , then f is called a *boundary function* for f if, and only if, for each $x \in E$ there exists an arc y at x such that $f(z) \rightarrow f(x)$ as $z \rightarrow x$ along y . Let S be another metric space. We shall say that a function f is of *Baire class 1* (S, M) if

- (i) domain $f = S$,
- (ii) range f and
- (iii) there exists a sequence of continuous functions, each mapping S' into M , such that $\langle f_n \rangle \rightarrow f$ pointwise on S .

We shall say that f is of *honorary Baire class 2* (S, M) if

- (i) domain $f = S$,
- (ii) range $f \subseteq A_f$, and
- (iii) there exists a countable set $N \subseteq S$ and there exists a function g of Baire class 1 (S, A_f) such that $f(x) = g(x)$ for every $x \in S - N$.

It is known that if f is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$ and any boundary function for f is of honorary Baire class ≤ 2 $(C, \text{Riemann sphere})$. (See^{1,2,3,4}, [9].) J. E. McMillan⁵ posed the following problem. If A is a given set in C of type $F_{\sigma\delta}$, and if f is a function of honorary Baire class ≤ 2 $(A, \text{Riemann sphere})$, does there always exist a continuous function g mapping D into the Riemann sphere such that A is the set of curvilinear convergence of g and g is a boundary function for f ? The purpose of this paper is to give an affirmative answer to McMillan's question. However, the corresponding question for real-valued functions remains open. (See Problems 1 and 2 at the end of this paper.) In proving our result, we first give a proof under the assumption that f is a bounded complexvalued function, and we then use a certain device to transfer the theorem to the Riemann sphere. As we shall indicate in an appendix, the same device can be

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used to transfer certain results concerning real-valued functions of the first Baire class to the case of functions taking values on the Riemann sphere.

¹ S. Banach, Über analytisch darstellbare Operationen in abstrakten Räumen, *Fund. Math.*, 17 (1931) 283-295.

² P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

³ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

⁴ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

⁵ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

Our proof is divided into several major steps, which are labeled (A), (B), (C), etc. The proofs of some of the major steps are divided into smaller steps, which are labeled (I), (II), (III), etc. The results (A) and (B) are taken from the author's doctoral dissertation⁶.

Throughout this paper we shall use the following notation. R denotes the set of real numbers, S^2 denotes the Riemann sphere, and R^n denotes n -dimensional Euclidean space. Points in R^n will be written in the form $\langle x_1, x_2, \dots, x_n \rangle$ (rather than (x_1, x_2, \dots, x_n)) in order to avoid confusion with open intervals of real numbers in the case $n = 2$. The empty set will be denoted by \emptyset . When we speak of a complex-valued function, we mean a function taking only *finite* complex values. The closure of a set E will be denoted either by \bar{E} or by $\text{Cl } E$. If I is an interval of real numbers, then I^* denotes the interior of I . If p is a point of some metric space and $r \in (0, \infty)$, then $S(r, p)$ denotes the set of all points of the space having distance (strictly) less than r from p .

We define

$Q = \{(x, y) \in R^2 : -1 \leq x \leq 1, 0 < y \leq 1\}$, $X = \{\langle x, 0 \rangle : -1 < x < 1\}$, $H = \{\langle x, y \rangle \in R^2 : y > 0\}$.

It will be convenient to identify $\langle x, 0 \rangle$ with the real number x , and X with $(-1, 1)$. If f is a complex-valued function defined in Q , then we shall understand the set of curvilinear convergence of f to mean the set of all $x \in X$ for which there exists an arc y at x (contained in the interior of Q except for its end point at x) such that f approaches a finite limit along y . If $a \in X$, $\epsilon > 0$, and $0 < \epsilon < 1$ then we let

$s(a, \epsilon, 0) = \{\langle x, y \rangle \in R^2 : 0 < y < \epsilon, a - y \cot \theta < x < a + y \cot \theta\}$.

Thus $s(a, \epsilon, 0)$ is the interior of an isosceles triangle in H with apex at a .

(A) If $A \cap X$ is a set of type $F_{\sigma\delta}$ then there exists a bounded continuous realvalued function g defined in Q such that

- (i) for each $x \in A$, $g(z) \rightarrow 0$ as z approaches x through $s(x, 1, 0)$, and
- (ii) if $x \in X$, and if there exists an arc y at x such that $g(z) \rightarrow 0$ as z approaches x along y , then $x \in A$.

(I) Let E_1 and E_2 be two sets on the real line. A point $p \in R$ will be called a *splitting point* for E_1 and E_2 if either

- (i) $x_1 \rightarrow p$ for all $x_1 \in E_1$ and $p \leq x_2$ for all $x_2 \in E_2$, or
- (ii) $x_2 \rightarrow p$ for all $x_2 \in E_2$ and $p \geq x_1$ for all $x_1 \in E_1$.

We will say that E_1 and E_2 *split* if and only if there exists a splitting point for E_1 and E_2 .

(II) By a *special family* we mean a family of subsets of X such that

- (i) \mathcal{F} is nonempty,
- (ii) \mathcal{F} is countable,
- (iii) each member of \mathcal{F} is compact,
- (iv) if $E, F \in \mathcal{F}$ then either $E = F$, $E \cap F = \emptyset$, or E and F split.

⁶ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

(III) If \mathcal{F} is an F_σ set, then there exists a special family \mathcal{A} such that $E = \bigcup_{n=1}^{\infty} A_n$.

Proof. We can write $E = \bigcup_{n=1}^{\infty} A_n$ where $A_i \cap A_j = \emptyset$, A_n is closed, and $A_n \subset A_{n+1}$ for all n . Observe that if I is any open interval contained in X , then there exists a countable family $\{J_n\}_{n=1}^{\infty}$ of compact intervals contained in X such that $I = \bigcup_{n=1}^{\infty} J_n$, and n/m implies that J_n and J_m split. Since $X - A_n$ is a countable disjoint union of open intervals, it follows that we can choose (for each n) a family

\mathcal{I}_n

of compact intervals such that $X - A_n = \bigcup_{j=1}^{\infty} I_{nj}$ and $I_{nj} \cap I_{mk} = \emptyset$ implies that I_{nj} and I_{mk} split. Let

$$\mathcal{A} = \{A_i\} \cup \{I_{nj} : n = 1, 2, \dots; j = 1, 2, \dots\}.$$

Then \mathcal{A} is a countable family of compact sets, and

$$E = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} (X - A_n) \cap I_{nj} = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} I_{nj} = E.$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} I_{nj} = E.$$

Let F_1 and F_2 be any two distinct members of \mathcal{A} . If either F_1 or F_2 is A_n , then F_1 and F_2 are automatically disjoint. If neither F_1 nor F_2 is A_n , then we can write

$$F_1 = \bigcup_{i=1}^{\infty} I_{1i}, \quad F_2 = \bigcup_{j=1}^{\infty} I_{2j}.$$

$$F_2 = \bigcup_{j=1}^{\infty} I_{2j} \subset \bigcup_{j=1}^{\infty} I_{1j} = F_1.$$

If $n(1) < n(2)$, then $n(1)+1 \leq n(2)$, so

$$F_2 \subset \bigcup_{j=1}^{\infty} I_{1j} = F_1, \quad \text{and therefore } F_1 \text{ and } F_2 \text{ are disjoint.}$$

If $n(1) = n(2)$, a similar argument shows that F_1 and F_2 are disjoint. Now suppose $n(1) > n(2)$. Then, since F_1/F_2 , we have $j(1)/j(2)$. So $j(1) \cdot n(2) < j(2) \cdot n(1)$ and $I_{1j(1)} \cap I_{2j(2)} = \emptyset$ split, and consequently F_1 and F_2 split. We have shown that any two distinct members of \mathcal{A} either split or are disjoint, so

\mathcal{A} is a special family.

(IV) Let $A \subset X$ be a set of type $F_{\sigma\delta}$. Then there exists a sequence of special families $\{\mathcal{A}_n\}_{n=1}^{\infty}$ such that

$$(i) \quad \bigcup_{n=1}^{\infty} \bigcup_{A \in \mathcal{A}_n} A = A,$$

$$(ii) \quad \text{if } I \text{ and } E \in \mathcal{A}_{n+1}, \text{ then there exists } F \text{ with } E \cap F = I.$$

Proof. There exist sets $A_i \subset \mathbb{R}^2$ such that $A = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_{ij}$. By (III), we can choose (for each n) a special family \mathcal{A}_n such that $\bigcup_{A \in \mathcal{A}_n} A = A_n$. Let For 1, let

$$\mathcal{A}_1 = \{F_n : n \in \mathbb{N}\}.$$

By induction on n , one can show that each \mathcal{A}_n is a special family and that $A_n = \bigcup_{A \in \mathcal{A}_n} A$ is clear that the other conditions are satisfied.

(V) Suppose that J is a nonempty interval with X , and let a, b ($a < b$) be the end points of J . By $\text{Trap}(J, e, \theta)$ (where $\theta \in (0, \pi/2)$ and $e > 0$) we mean the trapezoidshaped open set defined by

$$\text{Trap}(J, e, \theta) = \{x : 0 < y < e, a+y \cot \theta < x < b-y \cot \theta\}.$$

For $\theta \in (0, \pi/2)$ let $\text{Tri}(J, \theta)$ be the closed triangular area defined by

$$\text{Tri}(J, \theta) = \{x : y \geq 0, a+y \cot \theta \leq x \leq b-y \cot \theta\}.$$

If K is a nonempty compact subset of X , let $J(K)$ be the smallest closed interval containing K . If $e > 0$ and $0 < \delta < \frac{1}{2}$, then we define

$$B(K, e, \delta, 0) = \text{Trap}(\mathcal{L}, e, \delta) \cup \text{Tri}(Z, 0)_{\mathcal{L}}$$

where \mathcal{L} denotes the (possibly empty) set of disjoint nonempty open intervals whose union is $J(K) - K$.

We state without proof the following readily verifiable facts ((VI) through (XVIII)).

(VI) $s(x, e, 0)$ is an open subset of H .

(VII) $\text{Cl}[s(x, \delta, 0)] \cap y = \{x\}$.

(VIII) If $e < \delta$ and $0' < 0$, then $\text{Cl}[s(x, e, 0)] \cap H \cap s(x, \delta, 0')$.

(IX) If x/j and $c, 0$ are given, then there exists $\delta > 0$ such that, for every $\delta, j(x, e, 0)$ and $s(y, \delta, 0)$ are disjoint.

(X) $B(K, e, \delta, 0)$ is an open subset of H .

(XI) If K_1 and K_2 split, then, for any e, δ , and $0, B(K_1, e, \delta, 0)$ and $Z \cap C_2, \delta, 0)$ are disjoint.

(XII) If K_1 and K_2 are disjoint compact subsets of X and if $e, \delta, 0$ are given, then there exists $\delta > 0$ such that for every $\delta, B(K_1, e, \delta, 0)$ and $B(K_2, \delta, 0)$ are disjoint.

(XIII) $\text{Cl}[B(K, e, \delta, 0)] \cap X = K$.

(XIV) Suppose that $K \cap K, c > \frac{1}{2}$, and $0 < \delta < \frac{1}{2}$. Then $\text{Cl}[B(\delta, \delta, 0)] \cap H \cap B(K, \delta, 0)$.

(XV) Suppose that $a < 0$ and $x \in J(X)^*$. Then, for any e, δ and $0, B(K, e, \delta, 0)$ and $s(x, \delta, 0)$ are disjoint.

(XVI) Suppose that $x \in K$ and that $e, \delta, 0$ are given. Then there exists $\delta > 0$ such that for every $\delta, s(x, \delta, 0)$ and $B(K, e, \delta, 0)$ are disjoint.

(XVII) Suppose that $x \in K$ and that $\delta, a, 0, 0$ are given. Then there exists $s > 0$ such that for every $s(x, \delta, 0)$ and $B(K, \delta, a, 0)$ are disjoint.

(XVIII) Suppose that $x \in K \cap J(K)^*$ and $0 < \delta < \frac{1}{2}$. Let e be given. Then there exists $\delta > 0$ such that for every $\delta, \text{Cl}[s(x, \delta, 0)] \cap B(K, e, \delta, 0)$.

(XIX) If \mathcal{E} is a special family, let \mathcal{E}^2 be the set of all members of \mathcal{E} that have two or more points, and let $E(\mathcal{E})$ be the set of all end points of intervals $J(F)$ where $F \in \mathcal{E}$ and $f \neq 0$.

Suppose that $0 < \delta < \frac{1}{2}$, and that SF is a special family. By a *pair of special $a, f, 0$ functions* for I mean a pair (\mathcal{E}, δ) , where e and δ are positive real-valued functions, the domain of e is $E(\mathcal{E})$, the domain of δ is \mathcal{E}^2 , and

- (i) for each $\delta > 0$, there exist at most finitely many $F \in \mathcal{E}^2$ such that $\delta(F) > \delta$;
- (ii) for each $q > 0$, there exist at most finitely many $x \in E(\mathcal{E})$ such that $e(x) > q$;
- (iii) if $x, x' \in E(\mathcal{E})$ and $x \neq x'$, then $s(x, \mathcal{E}(x), 0)$ and $s(x', \mathcal{E}(x'), 0)$ are disjoint;
- (iv) if $F, K \in \mathcal{E}^2$ and $F \cap K$, then $B(F, \delta(F), a, f)$ and $B(K, \delta(K), a, f)$ are disjoint;
- (v) if $x \in E(\mathcal{E})$ and $F \in \mathcal{E}^2$, then $s(x, e(x), 0)$ and $B(F, \delta(F), a, f)$ are disjoint.

(XX) Let \mathcal{E} be a special family and suppose that $0 < \delta < \frac{1}{2}$. Then there exists a pair of special $a, f, 0$ functions for

Though a formal proof of this statement is lengthy, it requires no originality, so we omit the details. The idea is to arrange the members of \mathcal{E} in a finite or infinite sequence F_x, F_2, F_s, \dots , and then define e and δ inductively. One makes use of statements (IX), (XI), (XII), (XV), (XVI), (XVII).

(XXI) Let \mathcal{E} be a special family, and suppose $0 < \alpha < 1$. Let (\langle, δ) be a pair of special a, f, θ functions for \mathcal{E} . If e_1, δ_1 are two real-valued functions having domains $E(\mathcal{E})$ and \mathcal{E}^2 respectively, and if

$$0 < e_1(x) \leq \langle(x) \text{ for all } x \in E(\mathcal{E}'),$$

$$0 < \delta_1(F) \leq \delta(F) \text{ for all } F \in \mathcal{E}^2,$$

then (\langle, δ_1) is a pair of special a, f, θ functions for \mathcal{E} .

The proof of this statement is trivial.

(XXII) We now proceed to the proof of statement (A) itself. Let A be our given F_{00} set. By (IV), we can choose a sequence of special families such that $A = \bigcup_{i=1}^{\infty} (U_i \cap A)$, and for each $K \in \mathbb{N}$ there exists F_K with $K \in F_K$.

Let $\{j_n\}_{n=1}^{\infty}$ be a strictly increasing sequence in $(0, j_w)$ converging to j_r .

Let $\{a_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in (i_n, j_w) converging to i_n .

Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence in (i_n, j_w) converging to j_r .

Let $E_n = E(n_k)$.

Let (e_k, δ_k) be any pair of special a, f, θ functions for \mathcal{E}_k .

Now suppose that for each $k \in \mathbb{N}$ we have chosen a pair of special $\langle k, P_k, Q_k$ functions $(e(k, \bullet), \delta(k, \bullet))$ for in such a way that

(i) whenever $1 \leq k \leq n-1$, $x \in E_{k+1}$, $F_k \in F_k$, and $x \in F_k \setminus J(F)^*$, then

$$C_1[s(x, e(k+1, x), 0_{k+1})] \cap H \subseteq B(F, \delta(k, F), a_k, f_k);$$

(ii) whenever $1 \leq k \leq n-1$, $x \in E_{k+1}$, and $x \in E_k$, then

$$C_1[s(x, e(k+1, x), f_{k+1})] \cap H \subseteq s(x, e(k, x), 0_k);$$

(iii) whenever $K \in F_{n-1}$, $x \in (\langle_{n-1})^2$, $F_n \in F_n$, and $K \in F$, then

$$C_1[B(K, \delta(k+1, K), a_{k+1}, f_{k+1})] \subseteq B(F, \delta(k, F), a_k, f_k).$$

Then we construct $(s(n+1, \bullet), \delta(n+1, \bullet))$ as follows. Let (e, δ) be any pair of special $a_{n+1}, \mathcal{E}_{n+1}, \theta_{n+1}$ functions for \mathcal{E}_{n+1} . If $x \in E_{n+1} \setminus E_n$ then for some unique $F_n \in F_n$, $x \in F_n \setminus J(F)^*$. By (XVIII), we can choose $f(x) > 0$ so that $\langle(x) \leq f(x)$ implies

$$C_1[s(x, e(n+1, x), 0_{n+1})] \subseteq B(F, \delta(n, F), a_n, f_n)$$

We set $e(n+1, x) = \min \{c(x), f(x)\}$. On the other hand, if $x \in E_{n+1} \cap E_n$ then we set $e(n+1, x) = \min \{e(x), \langle(n, x)\}$.

If $K \in (\langle_{n+1})^2$, then there exists a unique $F_n \in F_n$ with $K \in F_n$. Set

$$\delta(n+1, K) = \min \{\delta(F), \delta(n, F)\}.$$

By (XXI), $(e(n+1, \bullet), \delta(n+1, \bullet))$ is a pair of special $a_{n+1}, \mathcal{E}_{n+1}, \theta_{n+1}$ functions for \mathcal{E}_{n+1} , and, by (VIII) and (XIV), conditions (i), (ii), (iii) are still satisfied when n is replaced by $n+1$. Thus we can inductively construct a pair $(e(n, \bullet), \delta(n, \bullet))$ of special $a_n, \mathcal{E}_n, \theta_n$ functions for in such a way that (i), (ii), and (iii) are satisfied for every n .

Let

$$U_n = \bigcup \{s(x_g, 8\{n_g, x\}, 0_n) \mid U \cap b(f_g, 8(w, f), a_n, p_n)\}. \quad L^{\text{ref}}_n \cap J \cap LFe(\wedge n)^2 \cap J$$

Then U_n is open. For fixed n_g all the various sets $s(x_g, e(n_g, x), 0_n)$ ($x \in F_n$) and $B(F, 8(n, F), a_n, 0_n)$ ($Fe(\wedge)^2$) are open and pairwise disjoint, so that every component of U_n is contained in one of the sets $s(x_g, e(n_g, x), 0_n)$ ($x \in F_n$) or $B(F, 8(n_g F)_g a_{n_g} p_n)$ ($Fe(\wedge_n)^2$). It therefore follows from (VII) and (XIII) that if W is any component of U_{n_g} then

(1)

From conditions (i) and (ii) in the definition of a pair of special a , \mathcal{L} , θ functions, it follows that

$$U_n \cap H = \bigcup \text{Cl} [s(x, e(n_g, x), 0_n)] \cap \bigcup [J \cap \text{Cl} [B(F, 8(w, F), a_n, \&)] \cap \bigcup.$$

$$l^*-G \wedge n \cap J \cap Lre(\wedge_n)^2 \cap J$$

Consequently, conditions (i), (ii), (iii) in our inductive construction of $(e(n_g, \bullet), 8(\langle, \bullet))$ (together with the fact that $x \in F_{n+1} \rightarrow E_n$ implies $x \in F \cap J(F)^*$ for some $Fe(\langle \wedge)^2$) imply that $U_{n+1} \cap U_n$ for every n .

By Urysohn's Lemma there exists a continuous function $g_n: H \rightarrow [0, 1]$ such that $g_n(z) = 1$ for $z \in H \setminus U_n$ and $g_n(z) = 0$ for $z \in U_{n+1}$. Let

$$g^{\mathcal{E}} = 2 \cap n = 1$$

Then $0 \leq g(z) \leq 1$, and the series converges uniformly, so g is continuous in H .

If $z \in U_{n_g}$ then $z \in U_m$ for every $m \geq n_g$ so that $1 = g_n(z) = g_{n+1}(z) = g_{n+2}(z) = \dots$, and hence

Also, if $z \in U_{B+1}$, then $z \in U_u \cap U_2, \dots \cap U_{n+1}$, so that $0 = g_i(z) = g_2(z) = \dots = g_n(z)$, and

$$(3) \quad g(z) \in \mathbb{R}^m = \mathbb{R}^n \cap (z \in U_{n+x}).$$

$$m = n + 1$$

Let $x_0 \in A$ be given. We must show that $g(z) \rightarrow 0$ as z approaches x_0 through $s(x_0, L \cap i^{77})$. Take any natural number n . Since $x_0 \in J^{77}_{+1}$, it follows that either $x_0 \in GF_{n+1}$ or else $x_0 \in F \cap J(F)^*$ for some $Fe(J_{n+1})^2$. In the first case, set $rj = e(n+1, x_0)$. In the second case, (XVIII) shows that we can choose $rj > 0$ small enough so that

$$s(x_0, rj) \subset B(F, 8(h+1, F), a_n + i, +$$

Suppose $\langle x, y \rangle \in s(x_0, 1, far)$ and $y < rj$. Then, in the first case,

$$\langle X, y \rangle \in s(x_0, rj) \supset Q \cap s(x_0, e(n+1, X_0), 0_{n+1}) \cap \wedge_{n+1},$$

and, in the second case,

$$\langle x, y \rangle \in s(x_0, rj, < 77) \subset B(F, 8(n+1, F), a_{n+1}, p_{n+1}) \subset U_{n+1}.$$

Thus, referring to (3), we see that $g(x, y) \rightarrow 0$ whenever $\langle x, y \rangle \in s(x_0, 1, 77)$ and $y < rj$. Therefore $g(z) \rightarrow 0$ as z approaches x_0 through $s(x_0, 1, 77)$.

Let X_i be a point of X , and assume there exists an arc y at x_Y such that $g(z) \rightarrow 0$ as z approaches x_{\pm} along y . Then y has a subarc y with one end point at X_i such that By (2), $\neg \{x^{\wedge \wedge \wedge}$. Therefore, by (1), $x_1 \in$

$U \cap \langle \wedge$. Since n is arbitrary,

$$*1 \quad 6 \quad A \cap (U \cap n) = A. \quad n = 1$$

Thus, by restricting g to Q we obtain the desired function.

(B) Let A be a subset of X of type F_{ad} , and let f be a bounded complex-valued function of honorary Baire class $\hat{2}(A, F^2)$. Then there exists a bounded continuous complex-valued function h defined in Q such that, for each $x \in A$, there exists an arc y at x with $y \rightarrow \{x\}^s(x, 1, \mathbb{R})$ and

$$\lim h(z) = \langle \mathcal{L}(x) \rangle.$$

(I) Let Z be a bounded open interval in \mathbb{R} , and let $f: Z \rightarrow \mathbb{R}$ be a bounded, strictly increasing function. Then there exists a continuous, weakly increasing function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $g(f(x)) = x$ for every $x \in Z$. (This result is probably not new, but I do not know of a reference for it, so I am obliged to prove it here.)

Proof. Let $Z = f(I)$, let $c = \inf Z$, and let $d = \sup Z$. Observe that $Z \subset (c, d)$, and that $f^{-1}: Z \rightarrow I$ is strictly increasing. I assert that for each $x \in (c, d)$

$$(4) \sup \{f(y) \mid (c, x] \cap Z \neq \emptyset\} = \sup \{f(y) \mid (c, x) \cap Z \neq \emptyset\}.$$

If $x \notin Z$, the equation is trivial. Suppose $x \in Z$. Then

$$c < y < f^{-1}(x) \Rightarrow (f(y) < x \text{ and } f(y) \in Z),$$

so that $(c, f^{-1}(x)) \cap Z \neq \emptyset$. Hence

$$\sup \{f(y) \mid (c, x) \cap Z \neq \emptyset\} = \sup \{f(y) \mid (c, x] \cap Z \neq \emptyset\}.$$

The opposite inequality is trivial, so (4) is established.

I also assert that for each $x \in (c, d)$

$$(5) \inf \{f(y) \mid (x, d) \cap Z \neq \emptyset\} = \inf \{f(y) \mid (x, d] \cap Z \neq \emptyset\}.$$

Obviously,

$$\inf \{f(y) \mid (x, d) \cap Z \neq \emptyset\} = \inf \{f(y) \mid (x, d] \cap Z \neq \emptyset\}.$$

Take any $y > \sup \{f(y) \mid (c, x] \cap Z \neq \emptyset\}$. If $f(y) \leq x$, then $f(y) \in (c, x] \cap Z$, and so $y \in K\{c, x] \cap Z$, a contradiction. Thus $f(y) > x$ and $f(y) \in (x, d) \cap Z$. Therefore $y \in \{f(y) \mid (x, d) \cap Z \neq \emptyset\}$, so that $\inf \{f(y) \mid (x, d) \cap Z \neq \emptyset\} \leq y$. In view of the choice of y , this implies that

$$\inf \{f(y) \mid (x, d) \cap Z \neq \emptyset\} = \inf \{f(y) \mid (x, d] \cap Z \neq \emptyset\},$$

and (5) is established.

Define g^* on (c, d) by

$$g^*(x) = \sup \{f(y) \mid (c, x] \cap Z \neq \emptyset\} \quad (x \in (c, d)).$$

It is clear that g^* is weakly increasing and that $g^*(f(x)) = x$ for every $x \in Z$. The continuity of g^* can easily be deduced from the equations

$$\sup \{f(y) \mid (c, x) \cap Z \neq \emptyset\} = g^*(x), \quad \inf \{f(y) \mid (x, d) \cap Z \neq \emptyset\} = g^*(x),$$

which are established as follows:

$$\sup \{f(y) \mid (c, x) \cap Z \neq \emptyset\} = \sup \sup \{f(y) \mid (c, y] \cap Z \neq \emptyset\} \mid c < y < x$$

$$= \sup \{f(y) \mid (c, x) \cap Z \neq \emptyset\}$$

$$= \sup \{f(y) \mid (c, x] \cap Z \neq \emptyset\}$$

$$= g^*(x),$$

$$\inf \{f(y) \mid (x, d) \cap Z \neq \emptyset\} = \inf \sup \{f(y) \mid (c, y] \cap Z \neq \emptyset\} \mid x < y < d$$

$$= \inf \inf \{f(y) \mid (y, J) \cap Z \neq \emptyset\} \mid x < y < d$$

$$= \sup \{f(y) \mid (c, x] \cap Z \neq \emptyset\} = g^*(x).$$

We now extend g^* to all of \mathbb{R} by setting

$$g^*(x) = \inf d \quad \text{if } x < c,$$

$$g^*(x) = \sup c \quad \text{if } x > d, \text{ and we are finished.}$$

(II) Suppose that M is a metric space and that $u: M \rightarrow R$ is a function having the following property. For every sequence $\{p_n\}$ of points of M , every $p \in M$, and every $y \in R \cup \{-\infty, +\infty\}$, if $p_n \rightarrow p$ and $u(p_n) \rightarrow y$ as $n \rightarrow \infty$, then $y \in R$ and $u(p) = y$. Under this hypothesis, u is continuous.

Proof. Let $\{p_n\}$ be any sequence of points in M converging to a point $p \in M$. We have only to show that $u(p) = \lim u(p_n)$. But suppose $u(p_n) \not\rightarrow u(p)$. Then there exists a subsequence $\{u(p_{n(k)})\}$ and there exists $y \in R \cup \{-\infty, +\infty\}$ such that $y \neq u(p)$ and $u(p_{n(k)}) \rightarrow y$ as $k \rightarrow \infty$. Since $p_{n(k)} \rightarrow p$ as $k \rightarrow \infty$, this contradicts our hypothesis.

(III) Let $A \subset (-1, 1)$ be of type $F_{\sigma\delta}$ and let g be a complex-valued function of Baire class 1 on A . Then there exists a sequence $\{g_n\}$ of continuous functions, each mapping A into \mathbb{C} such that for each $x \in A$, $g_n(x) \rightarrow g(x)$ as $n \rightarrow \infty$.

Proof. This can be proved in a more general context, as shown in⁸. For a quick proof of the special case stated above, we can refer to a theorem of Bagemihl and McMillan [1, Theorem 2], which tells us that there exist continuous real-valued functions f_1 and f_2 defined in H such that, for each $x \in A$, f_1 has angular limit $\operatorname{Re} g(x)$ at x and f_2 has angular limit $\operatorname{Im} g(x)$ at x . For each $x \in A$ set

$$g_n(x) = f_1(x) + if_2(x),$$

$$|g_n(x)| \leq |g(x)|$$

(IV) Now we proceed to the proof of statement (B). Let g be a function of Baire class 1 on $A \subset \mathbb{C}$ and let E be a (possibly empty) countable subset of A such that $g(x) \neq \lim_{y \rightarrow x} g(y)$ for each $x \in E$. Let N be an infinite countable set with $E \cap N = \emptyset$. Let w be a real-valued function defined on N such that $w(s) > 0$ for each $s \in N$ and

$$2^{-n} < w(s) < 2^{1-n} \quad \text{for } s \in N_n$$

For each $x \in X = (-1, 1)$, let $N(x) = \{s \in N : -1 < s < x\}$. Define f on $(-1, 1)$ by setting

$$f(x) = x + 2 \sum_{s \in N(x)} w(s)$$

Then f is a bounded, strictly increasing function on $(-1, 1)$, and $|f(x) - x| < 2^{1-n}$ for

1. By (I), there exists a continuous, weakly increasing function $f^*: \mathbb{R} \rightarrow \mathbb{R}$ such that $f^*(f(x)) = x$ for each $x \in (-1, 1)$.

Let

$$H_0 = \{ \langle x, y \rangle \in \mathbb{R}^2 : 0 < y$$

For fixed $\langle x, y \rangle \in H_0$,

$$u_y = f_x(l-y)u$$

$$|u_y| \leq y$$

is a strictly increasing continuous function of u that approaches $+\infty$ as $u \rightarrow +\infty$ and $-\infty$ as $u \rightarrow -\infty$. Consequently there exists precisely one number $u(x, y)$ that satisfies the equation

$$(6) \quad \langle \mathcal{G}, y \rangle = 0.$$

⁸ F. Hausdorff, Über halbstetige Funktionen und deren Verallgemeinerung, *Math. Z.*, 5 (1919) 292-309.

I assert that $u(x, y)$ is a continuous function on H_o . We show this by using (II). Suppose that $\langle x, y \rangle \in H_o$, $u_0 \in E \cup \{-\infty, +\infty\}$, $\{\langle x_n, y_n \rangle\} \in \mathcal{L}/f_0$, $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$, and $u(x_n, y_n) \rightarrow w_0$. If $u_0 = +\infty$, then, as $n \rightarrow \infty$,

$$x_n - (1 - y_n)u(x_n, y_n) \rightarrow x y_n,$$

and so

$$U\langle X_n, A \rangle - / * (X_n \sim (1 \sim \wedge X X_n, K)) + \langle \rangle,$$

$$\setminus z_n /$$

which contradicts (6). So $i/0 \wedge +\infty$, and a similar argument shows that $m_0 = \infty$.

Thus, by (6),

$$0 = \lim [u(x_n, K) - / * (X_n \sim \circ \sim y) u(x), n \rightarrow \infty] L \setminus y_n / J$$

-

Consequently $u_0 = u(x, y)$. By (II), u is continuous.

From (III), there exists a sequence $\{g_n\}$ of continuous complex-valued functions defined on R such that $g_n(x) \rightarrow 0(x)$ as $n \rightarrow \infty$ for each $x \in A$. For $n \geq 2$, define

$$h_0(x, y) = (y_n(n+1) - n)g_n(u(x, y)) + ((n+1) - y_n(n+1))g_{n+1}(u(x, y))$$

when $1/(72 - 4) \wedge y \geq 1/\infty$. Then h_0 is continuous on H_o . Let $\{s_n\}_{n=1}^\infty$ be all the elements of N , where w/z_n implies $s_n \wedge s_m$. Let

$$r_n = \inf / (x), \mathbf{X} > \mathbf{S}_n \text{ } I_n = \sup / (X) = f(s_n), \mathbf{x} < \mathbf{s}_n$$

$$Z_n = \text{if } s_n \in E,$$

$$z_n = 0 \text{ if } s_n \notin E.$$

Notice that $r_n - /_n > 0$. If x and y are real numbers, define $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. For $\langle x, y \rangle \in H_o$, set

$$A_n(x, y) = [(1 - ny) \vee 0] I(1$$

$$r_n + l_n - 2s_n + 2$$

$$\vee \circ z_n.$$

$$L \setminus 'n h$$

Then A_n is continuous in H_o . Observe that $A_n(x, y) = 0$ when $y \wedge 1/n$. Using this fact, it is easy to show that, if we set

$$\infty$$

$$hl = h_0 + 2 \boxtimes n, n = 1$$

then is defined and continuous on H_o .

Let p be any point of A . The line (7) passes through $\langle p, 0 \rangle$, and, since $|/(p) - p| < 2^{1/2} - 1 = \text{ctn fw}$, the part of this line which lies in H_o is contained in $s(p, \text{l.fw})$. We show that h_Y approaches along this line. By substituting $(f(p) - p)y + p$ for x in the expression for $A_n(x, y)$, one obtains

$$A_n(x, y) = [(1 - ny) \vee 0]$$

$$(8) \quad 17$$

$$r_n + 4 + 2(-$$

$$?n-$$

If $P \wedge s_n$, then $f(p) \wedge l_n$, and one can verify directly that (8) vanishes. If $p > s_n$, then $f(p) r_n$, and again one can verify directly that (8) vanishes. Thus $A_n(x, y)$ vanishes along that part of the line (7) which lies in H .

Solving (7) for $f(p)$, we find that, along the given line,

$/ (p) = (x - (l - y)p)ly$, and hence

$p = / (\boxtimes(?)) = / ((x - (i - y)p)ly) -$.

Therefore (if $0 < y^{\wedge} |$) $p = u(x, y)$. Hence, if $\langle x, y \rangle$ satisfies (7), $n^{\wedge} 2$, and $1/(n+1) \ 1/n$, then

$h_0(x, y) = (yn(n+1) - n)g_{,,}(p) + ((\gg +!) - yn(n+1))g_{n+j}(p)$,

so that $A_0(x, y)$ lies on the line segment joining $g_n(p)$ to $g_{n+i}(p)$. It follows that $/z_0(x, y)$ approaches $0(p)$ as $\langle x, y \rangle$ approaches p along the line (7). Since each A_n vanishes on the part of this line lying in H , $h_r(x, y)$ also approaches $\wedge(p)$ along this line.

Let s_m be any point of E . The definition of f shows that

$|/(x) - x| \ 2^w(5)$

seN

for all x , and from this it easily follows that

$K - Jml \ 2 \ 1^{m \ll J} ml = 2$

$sgN \ sgN$

Hence

So the part of the line

(9)

that lies in H_o is contained in $s(s_m, 1, fir)$. We show that approaches $\wedge(s_m)$ as $z \rightarrow s_m$ along this line. Substituting the value of x given by (9) into the expression for A_n , we obtain

$A_n(x, y) = [(1 - ny) \vee 0]$

(10)

\boxtimes_n

If $s_m < s_B$, then $/_m < r_m \mathcal{L} /_n < r_n$, and one can verify that (10) vanishes. If $s_n < s_m$, then $l_n < r_n \mathcal{L} l_m < r_m$, and again one can verify that (10) vanishes. Thus, for n/m , $A_n(x, y) = 0$ when $\langle x, y \rangle$ lies on the line (9) and in H .

If we take $n = m$ in (10), we obtain

$A_m(x, y) = [(1 - wy) \vee 0]z_m$.

Therefore $A_m(x, y)$ approaches $z_m = \wedge(s_m) - \wedge(s_m)$ along the given line.

Take any $\langle x, y \rangle \in H_o$ satisfying (9), and take any a and b satisfying

(11) $a < s_m < b$.

Then $f(a) \wedge l_m < |(r_m + /_m) < r_m \ \mathcal{L} f(b)$, so that

$(f(a) - s_m)y + s_m < x < (f(b) - s_m)y + s_m$;

from which we deduce that

$/ (a) < (x - (1 - y)s_m)ly < f(b)$.

Since f^* is weakly increasing,

$a = \mathcal{L} f^*((x - (l - y)s_m)ly) \ \mathcal{L} f^*(f(b) \gg) = b$.

Because a and b were taken to be any two numbers satisfying (11), we conclude that

$$s_m = f^*(.(x-(l-y)s_m)/y),$$

whence it follows that $u(x,y)=s_m$. Thus

$$h_0(x, y) = (yn(n+1) - ri)g_n(s_m) + ((\ll +!) - yn(n+1))g_{n+1}(s_m)$$

when $1/(n+1) \wedge y \wedge 1/n$. Consequently $h_0(x,y)$ approaches $\wedge(s_m)$ along the line (9); so $Ai(x, y)$ approaches along the given line.

We have shown that, for each $x \in A$, there exists a line segment at x , lying in $s(x)$, such that $A_1(z) \wedge \wedge(x)$ as $z \rightarrow x$ along the line segment. We do not know that h_Y is bounded, but this is easily patched up. Choose a real number B such that, for all $x \in A$,

$$-B < \operatorname{Re} \wedge(x) < B, \quad -B < \operatorname{Im} \wedge(x) < B, \quad \text{and set}$$

$$h(z) = ([(\operatorname{Re} A_1(z)) \vee (-B)] \wedge B) + i([(\operatorname{Im} f_1(z)) \vee (-B)] \wedge B).$$

If we extend h to a bounded continuous function defined in H , and then restrict h to Q , we have the desired function.

(C) Let $d\{t\}$ be a weakly increasing, positive, real-valued function defined for $0 < t \leq 1$. Then there exists a continuous, complex-valued function k defined in Q , with $|\wedge(z)| \leq 2^{1/2}$ for all $z \in Q$, such that for each $a \in (0, 1]$ and for each arc

$$y \in \{ \langle x, y \rangle : -1 \leq x \leq 1, 0 < y \leq a \},$$

$$\{ \text{diameter } y \} \leq d(a) \text{ implies } \{ \text{diameter } k\{y\} \} \leq 2.$$

Proof. Let $p\{x\} = \int_0^x d\{t\} dt$ ($0 < x \leq 1$). Then p is positive, continuous, and strictly increasing, and $p\{x\} \leq d\{x\}$. Let $a \in (0, 1]$ be given. Since $p\{x\} Y^r$ is uniformly continuous on each compact subset of $(0, 1]$, there exists $e \in (0, 1]$ such that

$$(|a| \leq 1 \text{ and } |x_1 - x_2| < e)$$

implies

$$|X^*1|^{-1} \sim P^*(2)^{-1} |1 -$$

Let $e\{a\}$ be the supremum of all such e . Then $e\{a\}$ is an increasing function of a , and

$$|x_1 - x_2| \leq 1 \text{ and } |x_1 - x_2| < e(a))$$

implies

Set $q(x) = \int_0^x e(t) dt$. Then q is positive, continuous, and strictly increasing, and $q(x) \leq d(x)$. Let $m\{x\} = \min \{p(x), q(x)\}$. For $\langle x, y \rangle \in Q$, define

$$k\{y\} = \sin \{2n/ym\{y\}\}, \quad k_2\{x, y\} = \sin \{4n x l p\{y\}\},$$

$$k\{x, y\} = i(y) + i_2(x, y).$$

Now suppose that $a \in (0, 1]$ is given, and suppose that $y \in \{ \langle x, y \rangle : -1 \leq x \leq 1, 0 < y \leq a \}$ is an arc with $(\text{diameter } y) \leq d(a)$. Choose $z_1 = \langle x_1, y_1 \rangle$ and $z_2 = \langle x_2, y_2 \rangle$ in y so that $|z_1 - z_2| \leq d\{a\}$. Assume without loss of generality that $y_2 \geq y_1$. We can choose a' so that $0 < a' \leq y_1 \leq a' \leq a$. Since $m\{a'\} \leq d\{a'\} \leq d\{a\}$, and since $|z_1 - z_2| \leq d\{a\}$, we must have either

$$(12) \quad |y_1 - y_2| \leq m\{a'\}$$

or

$$(13) \quad |x_1 - x_2| \leq m\{a'\} \text{ and } |x_1 - x_2| \leq d\{a'\}.$$

First assume that (12) holds. Here $m\{y_2\} \leq m\{y_1\} \leq m\{a'\}$, so

$$2r/y_1 m(y_1) \leq 2r/y_2 m(y_2),$$

may assume without loss of generality that y has an end point in $\{ \langle x, 1 \rangle : -1 \leq x \leq 1 \}$. By the properties of g , it will be enough to show that g approaches zero along y . Assume that g does not approach zero along y . Then there exists $\epsilon \in (0, 1]$ and there exists a sequence $\{z_n\}$ such that $z_n \in y - \{x\}$, $z_n \rightarrow x$ as $n \rightarrow \infty$, and $|g(z_n)| \geq \epsilon$ for all n . Write $z_n = (x_n, y_n)$. Choose N so that $n \geq N$ implies $y_n <$

For the time being, let n be a fixed integer greater than or equal to N . Set $a = 4y_n/3$. Let y be the component of $y \cap \text{Cl}[S(J(tz), z_n)]$ that contains z_n . (Recall that $S(d(a), z_n) = \{z : |z - z_n| < d(a)\}$.) Then

$$d(a) \text{ diameter } / 2 \leq d(a),$$

and, since $d(a) \leq la$,

$$7 \leq \{ \langle x, y \rangle : la \leq y \leq a \}.$$

By the choice of k , there exist points p and q in y with $| \mathcal{L}(\cdot) - \mathcal{L}(\cdot) | \leq 7$. We have $lp - q \leq 2d(a) < dfla$, so, by the definition of $J(t)$,

$$|A(p) - A(q)| \leq la < |e|.$$

Similarly,

$$|g(p) - g(z_n)| \leq la < |e|,$$

$$|g(q) - S(O)| \leq la <$$

Thus

$$I/(P) - (9) \leq |g(p)k(p) - g(q)k(q)| \leq |h(p) - h(q)|$$

$$I \leq |g(p)k(p) - g(z_n)k(p) + g(z_n)k(p) - g(z_n)k(q) + g(z_n)k(q) - g(q)k(q)| \leq |g(z_n)| \\ k(p) - f_c(?) \leq k(O) \leq |g(P) - g(z_n)| \leq |\mathcal{L}(?)| \leq |g(\cdot) - g(z_n)| \leq le$$

$$Z \leq 2e - 2^{l_2}le - 2^{l_2}le - l_e > e.$$

Note that $|p - z_n| \leq d(a) \leq la = ly_n$, and similarly $|z_n - y_n| \leq y_n$.

We have now shown that, for each $n \geq N$, there exist points $p_n, q_n \in y$ with $|a - z_n| \leq ly_n$, $k_n - z_n \leq Hy_n$, and $|/(p_n)| \leq \epsilon$. But then $p_n \rightarrow x$ and $q_n \rightarrow x$ as $n \rightarrow \infty$, so f does not approach a limit along y . This is a contradiction. We conclude that $g(z) \rightarrow 0$ along y , and hence that $x \in A$.

(E) Let $A \subset C$ be a set of type F_{ad} , and let $\langle f \rangle$ be a bounded function of honorary Baire class $\leq 2(A, B^2)$. Then there exists a bounded continuous complex-valued function f defined in D such that A is the set of curvilinear convergence of f and $\langle f \rangle$ is a boundary function for f .

Proof. If $A = \emptyset$, this is trivial. If $A \neq \emptyset$, then we can assume, by making a suitable rotation of the disk, that $\langle 1, 0 \rangle \in A$. Let $G = D - S(\mathcal{E}, \langle 1, 0 \rangle)$ and let $L = C - \{ \langle 1, 0 \rangle \}$. Because $Q \cup X$ is homeomorphic with $G \cup \mathcal{L}$, we see from (D) that there exists a bounded continuous complex-valued function f_Y defined in G such that

(i) $A \cap L$ is the set of all points $x \in L$ such that f_Y approaches a limit along some arc at x , and

(ii) the restriction of $\langle f_Y \rangle$ to L is a boundary function for f_Y .

Since G is closed relative to Z , we can extend f_Y to a bounded continuous function f defined in D in such a way that f has $\langle f \rangle \equiv \langle 1, 0 \rangle$ as a radial limit at $\langle 1, 0 \rangle$. This f will have all the desired properties.

(F) Let S^2 denote the Riemann sphere, let $A \wedge C$ be a set of type F_{06} , and let \langle / \rangle be a function of honorary Baire class $\wedge^2(A, S^2)$. Then there exists a continuous function $f: D \rightarrow S^2$ such that A is the set of curvilinear convergence of f and \langle / \rangle is a boundary function for f

(1) We suppose that

$$S'^2 = \{ \langle x, y, z \rangle \in K^3 : x^2 + y^2 + z^2 = 1 \}.$$
 We let

$$U = \{ \langle x, y, z \rangle \in S^2 : -1 \leq z \leq 1 \}$$

$$V = \{ \langle x, y, z \rangle \in S^2 : -1 \leq z \leq 1 \}$$

$$Z_u = \{ y, z \in S^2 : -1 \leq z \leq 1 \},$$

$$Z_v = \{ \langle x, y, z \rangle \in S^2 : -1 \leq z \leq 1 \}$$

We define mappings $0^*: Z_v \rightarrow U$ and $\langle J \rangle_y: Z_v \rightarrow V$ by setting

$$\langle Mx, y, z \rangle = \langle x(4z^2 - 1), y(4z^2 - 1), z(4z^2 - 3) \rangle \ll x, y, z \rangle \in Z_u$$

and

$$\langle D_y(x, y, z) = \langle x(4z^2 - 1), y(4z^2 - 1), z(4z^2 - 3) \rangle \ll x, y, z \rangle \in Z_y).$$

Then 0^* is a one-to-one continuous function from Z_u onto U . Since Z_a and U are each homeomorphic to the unit disk D , it follows from [7, Corollary 1, p. 122] that $\langle \rangle /$ is a homeomorphism of Z_v onto U . Similarly, O_y is a homeomorphism of Z_y onto V ,

We define a continuous function $O: S^2 \rightarrow S^2$ by setting

$$O(x, y, z) = \langle D_u(x, y, z), -z \rangle,$$

$$O^*(x, y, z) = \langle x, y, -z \rangle, z$$

$$\langle D(x, y, z) = \langle D_v(x, y, z), -1 \leq z \leq 1 \rangle$$

Notice that for each $p \in S^2$, the inverse image set $h^{-1}(\{p\})$ contains at most three points.

(II) Most of the results of Hausdorff⁹ on real-valued Baire functions can easily be shown to hold also for functions taking values in A . We shall make free use of these results in this more general form.

(III) Now we proceed to the proof of (F). Let N be a countable subset of A such that the restriction of 0 to $A - N$ is of Baire class $\wedge(A - N, S^2)$, and let $A = \wedge - N$. It will be convenient to let $F_a(A_i)$ denote the class of all subsets of A_i that are of type F , relative to A_i and $G_i\{A\}$ the class of all subsets of A_x that are of type G relative to A . Since U and V are open subsets of S^2 and $U \cup V = S^2$, we see that $A_{\pm} \cap j^{-1}(U) \in F \wedge A \wedge, A, -/^{-1} \wedge \in G \wedge AJ$, and $A_1 - /^{-1}(F) \wedge A_1 \cap 0^{-1}(G)$. An elegant theorem of Sierpinski [8] now enables us to choose a set $X \in F_{\omega}(A_1) \cap G_d(A)$ such that

$$A - 0^{-1}(F) \cap K \in A_i \cap \wedge(U).$$

Let $L = A_i - K$. Then $L \in F \wedge AJ \cap G_f(A_i)$. Moreover, $\langle / \rangle(K) \cap U$ and $0(L) \in V$. Let $\langle / \rangle! = \langle !, 0, 0 \rangle$, and define $0: A \rightarrow S^2 - \{p\}$ as follows. Set

$$\wedge(x) = \langle D^{\wedge}(0(x)), \wedge(x) =$$

$$x \in K, x \in L.$$

⁹ F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

If $x \in N$, we let $0(x)$ be any element of $Z_V K J Z_V$ for which $d(0(x)) = 0(x)$. This choice of $0(x)$ is always possible, because $d(Z_Y \cup Z_Y) \supset U \cap V = S^2$. Let 0 be the restriction of 0 to $A \setminus K \cap L$. I assert that 0_0 is of Baire class

$1(A \setminus \{p_1\})$. Since $S^2 - \{p_1\}$ is homeomorphic to R^2 , it will suffice to show that $0_0 \setminus G \in F^1(A)$ for every open set $G \subset S^2 - \{p_1\}$. But

$$\begin{aligned} 0_0 \setminus G &= A \cap i/r \setminus G = [K \cap G] \cup [L \cap G] \\ &= [K \cap fW \cup Zu \cap G] \cup [L \cap f \setminus \bigwedge Z_v \subset G] \in F^1, \end{aligned}$$

so 0_0 is of Baire class $1(A, S^2 - \{p_1\})$. Now, $A \setminus A \setminus N$ is of type $G_{\delta\sigma}$ relative to A , so (again using the fact that $S^2 - \{p_1\}$ is homeomorphic to R^2) we can extend 0_0 to a function 0_L of Baire class ≤ 1 on $A \setminus \{p_1\}$. The existence of 0_L shows that 0 is of honorary Baire class $\leq 2(A, S^2 - \{p_1\})$. The range of 0 is contained in $Z \cup V$, so that the values of 0 are bounded away from p . Thus, if we still think of $S^2 - \{p_1\}$ as corresponding to the plane R^2 , 0 corresponds to a bounded function. By (E), there exists a continuous function $f_1: D \setminus \{p_1\} \rightarrow R^2$ such that the values of f_1 are bounded away from p . A is the set of curvilinear convergence of f_1 and 0 is a boundary function for f_1 . Let f denote the composite function $0 \circ f_1$. Then f is continuous and $0 = 0$ is a boundary function for f . It only remains to show that if x is a point of the set of curvilinear convergence of f , then $x \in A$. Let y be an arc at x along which f approaches a limit, and let $C(f|_y)$ denote the cluster set of $f|_y$ along y . Assume that $x \notin A$. Then 0 does not approach a limit along y , so $C(f|_y)$ contains infinitely many points. Now, 0 maps at most three points to any one given point, so $0(C(f|_y))$ contains infinitely many points. But $0(C(f|_y))$ is the cluster set of $0 \circ f|_y$ along y , and hence f does not approach a limit along y , contrary to our assumption. We conclude that $x \in A$ after all. This completes the proof of the theorem.

The following questions remain open.

Problem 1. If A is an arbitrary set of type $F_{\delta\sigma}$ in C , does there necessarily exist a continuous real-valued function in D having A as its set of curvilinear convergence?

Problem 2. If $A \subset C$ is a set of type $F_{\delta\sigma}$, and if 0 is a function of honorary Baire class $\leq 2(A, R)$, does there necessarily exist a continuous real-valued function in D having A as its set of curvilinear convergence and 0 as a boundary function?

Appendix. Some theorems concerning functions of Baire class 1 which take values on the Riemann sphere can be obtained by the technique used to prove (F). We use the notation set up in the proof of (F).

Theorem (a). Let M be a metric space, and let $\langle \cdot \rangle : M \rightarrow S^2$ be a function such that $\langle G \rangle$ is an F_a set for every open set $G \subset S^2$. Then $\langle f \rangle$ is of Baire class 1 (Af, S^2).

Proof. Since U and V are open and $U \cup V = S^2$, it follows that the set is F_a , the set $M \setminus G_\delta$, and By the theorem of Sierpinski [8], there exists a set K that is simultaneously F_a and G_δ such that $L = M \setminus K$. Then L is simultaneously F_a and G_δ , and $W \cup V$.

Define $0 : M \rightarrow S^2 \setminus \{p_j\}$ (where $\langle \cdot \rangle = \langle 1, 0, 0 \rangle$ by setting $\langle x \rangle = 0 - i(\langle x \rangle)$, $x \in K$, $0(x) = \langle D_y - \langle x \rangle \rangle$, $x \in L$).

If G is an open subset of $S^2 \setminus \{h\}$, then $= [KhWUZij G) \cup [L \cap n G))$,

so $0^{-1}(G)$ is an $F_{\sigma\delta}$ set. Since $S^2 \setminus \{p_1\}$ is homeomorphic to the plane, it follows that there exists a sequence $\{ip_n\}$ of continuous functions, each mapping M into $S^2 \setminus \{p_1\}$, such that pointwise on M . But then $0(0_n(x))0(\langle x \rangle) = \langle x \rangle$ for each fixed $x \in M$, so $\langle \cdot \rangle$ is of Baire class 1 (M, S^2).

A special case of Theorem (b) was proved (in effect) in [6, proof of Theorem 6] by means of a rather messy lemma (Lemma 3). Theorem (a) provides a proof that is both more general and more esthetically satisfactory.

Theorem (b). Let M be a metric space, and let $\langle \mathcal{L} : M \rightarrow S^2$ be a function. Then $\langle f \rangle$ is of honorary Baire class $\hat{2}(M, S^2)$ if, and only if, there exists a countable set $N \subset M$ such that, for every closed set $F \subset S^2$, $\langle \cdot \rangle^{-1}(F) \setminus N$ is a G_δ set.

Proof. The implication in one direction is trivial. Now assume that N is countable and that $M \setminus N$ is a G_δ set for every closed set $F \subset S^2$. Let $\langle \cdot \rangle_0$ be the restriction of $\langle \cdot \rangle$ to $M \setminus N$. Since S^2 is a subset of R^3 , $\langle \mathcal{L}_0$ is of Baire class $\hat{1}(M \setminus N, R^3)$. Because $M \setminus N$ is a G_δ set, $\langle \cdot \rangle_0$ can be extended to a function of Baire class

$\hat{1}(M, A^3)$. Now, $\langle x \rangle \in S^2$ except for only countably many x , so there exists some point q in the open ball enclosed by S^2 such that q is not in the range of f_a . Define a mapping $P : R^3 \setminus \{q\} \rightarrow S^2$ as follows. If $a \in R^3 \setminus \{q\}$, let L be the ray with end point at q which passes through a , and let $P(a)$ be the intersection point of L with S^2 . Then P is continuous and $P(a) = a$ for each $a \in S^2$. Let $\langle \cdot \rangle_0 \subset \langle \cdot \rangle$. If $\langle \cdot \rangle_0^{-1}(G) \setminus N$ is open, then $0^{-1}(G) = \langle \cdot \rangle^{-1}(G) \cup \langle \cdot \rangle_0^{-1}(G)$, so that $0^{-1}(G)$ is an $F_{\sigma\delta}$ set. Thus, by Theorem (a), $\langle \cdot \rangle$ is of Baire class 1 (M, S^2). Moreover, if $x \in N$, then $\langle x \rangle_0 = \langle x \rangle$, so that $0(x)P(\langle x \rangle) = \langle x \rangle$. Therefore $\langle \cdot \rangle$ is of honorary Baire class $\hat{2}(M, S^2)$.

An alternative proof of Theorem (b) could be given by combining Theorem (a) with the following result.

Theorem (c). Let M be a metric space, E a G_δ set in M , $\langle f \rangle$ a function of Baire class 1 (\mathcal{L}, S^2). Then $\langle \cdot \rangle$ can be extended to a function of Baire class 1 (Af, S^2).

To prove this, use the technique appearing in the proof of Theorem (a).

Finally, we note that a theorem proved by Bagemihl and McMillan for realvalued functions [1, Theorem 2] can be transferred to the Riemann sphere by means of our technique.

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9. July 1969 - Boundary functions and sets of curvilinear convergence for continuous functions

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Kaczynski, T.J. 1969. Boundary functions and sets of curvilinear convergence for continuous functions. *Trans. Am. Math. Soc.* 141:107-125.

MR0243078 Kaczynski, T. J. Boundary functions and sets of curvilinear convergence for continuous functions. *Trans. Amer. Math. Soc.* 141 1969 107.125. (Reviewer: J. E. McMillan) 30.62

Explanation by John D. Bullough

The author completes the investigation, initiated by Bagemihl and Piranian, of boundary functions of continuous complex-valued functions defined in the open unit disk D . the set of curvilinear convergence A of such a function f is defined to be the set of those e^{iT} at which f has a finite or infinite limit along some open Jordan arc lying in the disk and having one endpoint at e^{iT} . A boundary function of f is a function t defined on A such that each $t(e^{iT})$ is one of these limit values. The author proved that t differs from some function of the first Baire class at at most countably many points, and McMillan proved that A is of type $F(\sigma)$. By means of an intricate construction, the author proves that for any set A on the unit circle of type $F(\sigma)$, and for any function t defined on A that differs from some function of the first Baire class at at most countably many points, there exists a continuous complex-valued function f defined in D having A as its set of curvilinear convergence and having t as its boundary function. The author points out the the problem remains open for real-valued functions.

Article by Ted

10. Nov 1969 - The Set of Curvilinear Convergence...

Original PDF: 10. Nov 1969 - The Set of Curvilinear Convergence....pdf

Kaczynski, T.J. 1969. The set of curvilinear convergence of a continuous function defined in the interior of a cube. *Proc. Am. Math. Soc.* 23:323-327.

MR0248339 Kaczynski, T. J. The set of curvilinear convergence of a continuous function defined in the interior of a cube. *Proc. Amer. Math. Soc.* 23 1969 323.327. (Reviewer: J. E. McMillan) 30.62

Explanation by John D. Bullough

The set of points of the unit circle at which a continuous complex-valued function in the open unit disk has limits along curves (asymptotic values) is of type $F(\sigma)$ and, in general, has no other properties. The author shows that for continuous complex-valued functions defined in a cube, this set of "curvilinear convergence" does not even need to be a Borel set. He asks whether such an example can be given for real-valued functions.

Article by Ted

THE SET OF CURVILINEAR CONVERGENCE OF A CONTINUOUS FUNCTION DEFINED IN THE INTERIOR OF A CUBE

T. J. KACZYNSKI

Let Q be an open connected set in a finite-dimensional Euclidean space, and let f be a function mapping Q into another finite-dimensional Euclidean space. We define the *set of curvilinear convergence of f* to be

$\{p \in \text{boundary of } Q: \text{there exists a simple arc } \gamma \text{ with one endpoint at } p \text{ such that } \gamma \text{ — } \{p\} \text{ CQ and } f(\gamma) \text{ converges to a finite limit as } v \rightarrow p \text{ along } \gamma\}.$

J. E. McMillan¹ has shown that if Q is an open disk in the plane and if f is continuous in Q , then the set of curvilinear convergence of f is of type F_σ . In this paper we prove that there exists a bounded continuous complex-valued function f , defined in

¹ M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1961.

the interior of a three-dimensional cube, such that the set of curvilinear convergence of f is not a Borel set. Thus McMillan's theorem does not generalize to three dimensions. However, the following question remains open.

Problem. Does there exist a continuous real-valued function f , defined in the interior of a three-dimensional cube, such that the set of curvilinear convergence of f is not a Borel set?

Let

R be the set of real numbers

R^n — n -dimensional Euclidean space

$Q = \{(x, y) \in R^2 : 0 < y < 1 \text{ and } 0 < x < 1\}$

$\bar{Q} = \{(x, y, z) \in R^3 : 0 < y < 1, 0 < x < 1, \text{ and } 0 < z < 1\}$

Q° = interior of Q

K° = interior of K .

Let Q again represent an open connected subset of R^n . If $f: Q \rightarrow R^m$ is a function, we shall say that $a \in R^m$ is an *asymptotic value* of f iff there exists a continuous function $v: [0, 1] \rightarrow R^m$ such that $\text{dist}(f(t), a) \rightarrow 0$ and $f(t) \rightarrow a$ as $t \rightarrow 1$. (Note that a limit approached by f along a path which tends to ∞ may or may not be an asymptotic value by our definition.) We say that a is a *point asymptotic value* of f (at p) iff v can be chosen so that, as $t \rightarrow 1$, $v(t)$ approaches

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a point $a \in R^m$. Because of the result of [8], the set of curvilinear convergence of f is $\{p \in R^n : \text{there is a point asymptotic value of } f \text{ at } p\}$.

Lemma. There exists a continuous complex-valued function s defined in

$\{(x, y) \in R^2 : y > 0\}$,

with $|s(x, y)| < 1$ for all x and y , such that s has the following property.

Let E be the set of all asymptotic values of s that are real and lie in the interval $(-1, 1)$. Then E is equal to the set of all point asymptotic values of s that are real and lie in $(-1, 1)$, and E is not a Borel set.

Proof. Let A be an analytic subset of R that is not a Borel set. (This exists [7, p. 254].) We see from the paper of Kierst² that there exists a holomorphic function h defined in $\{z : z \text{ is a complex number and } |z| < 1\}$ such that h omits the three values $-i, i, \infty$ and $A \setminus \{-i, i\}$ is the set of all (finite) asymptotic values of h . The function h is then normal [5, p. 53], so, as pointed out by McMillan [6, p. 311], it follows from Theorem 1 of³ that $A \setminus \{-i, i\}$ is just the set of all (finite) point asymptotic values of h . We now obtain the desired function by setting

$$s(x, y) = \frac{h((1-y)e^{ix})}{1-y}$$

² P. T. Church, Ambiguous points of a function homeomorphic inside a sphere, *Michigan Math. J.*, 4 (1957) 155-156.

³ F. Bagemihl & G. Piranian, Boundary functions for functions defined in a disk, *Michigan Math. J.*, 8 (1961) 201-207.

Remark. Since the theorem we want to prove has nothing to do with meromorphic functions, it is unfortunate that the proof of the lemma depends on the theory of meromorphic functions. This can be avoided. The lemma can be proved by using [7, Theorem 113, p. 216], [1, Theorem 2, p. 179], and the methods of⁴, but this involves a messy construction, so we omit the details.

Proof. Let δ and E be as described in the lemma, and set $g(x, y) = s(x/y, y)$ for $(x, y) \in GQ$. The reader can verify that E equals the set of all real point asymptotic values of g at the point $(0, 0)$ which lie in the interval $(-1, 1)$. For each $t \in (0, 1]$, define $Z_t = \{ (x, y) \in GQ : |x| \leq t, |y| \geq 1/t \}$. Then $Z_t \subset E$ and $|Z_t| \rightarrow 0$ as $t \rightarrow 0^+$. For each $t \in (0, 1]$, let $d(t) = \inf\{|g(x, y) - g(x', y')| : (x, y), (x', y') \in Z_t, (x, y) \neq (x', y')\}$. Then $d(t) \rightarrow 0$ as $t \rightarrow 0^+$.

7 C $\{(x, y): -1 \leq x \leq 1 \text{ and } 0 < y \leq a\}$, $(\text{diameter } 7) \wedge d(a)$ implies $(\text{diameter } \mathcal{L}(7)) \wedge 2$.

$$\begin{aligned} & \text{I/O', } y', z') - f(x'', y'', z'') | \\ & = | (g \ll y') - z') k(x', y') - (g(x, y) - z) k(x', y') + (g(x, y) \sim \sim k(x', y') - (g(x, y) - \\ & z) k(x'', y'') + (g(x, y) - z) k(x'', y'') - (g(x'', y'') - z'') k(x'', y'') | \text{I } g(x, y) - z \text{ I } k(x', y') - \\ & k(x'', y'') \text{ I} \end{aligned}$$

Let $Z = \{(0, 0, 2) : -1 < \alpha < 1\}$, and let T be the set of curvilinear convergence of f . We wish to show that $rP \setminus L = \{(0, 0, 2) : 2G^+\}$.

⁵ S. Banach, Über analytisch darstellbare Operationen in abstrakten Räumen, *Fund. Math.*, 17 (1931) 283-295.

Suppose $\&G\mathcal{L}$ - Then there is an arc γ with one endpoint at $(0, 0)$ such that $\gamma \setminus \{(0, 0)\} \subset \mathbb{C}Q^\circ$ and g approaches b along γ . Let

$$\gamma' = \{(*, y, b) : (x, y) \in \gamma\}.$$

Then $g(x, y) \rightarrow 0$ as $(x, y, 2) \rightarrow (0, 0, b)$ along γ' . Thus, since k is bounded, $g(x, y, 2) \rightarrow 0$ along γ' , so $(0, 0, \&)GLHL$.

Now let us assume, conversely, that $(0, 0, \&)GrO\mathcal{L}$ and deduce that $\&G\mathcal{L}$ - Let γ' be an arc with one endpoint at $(0, 0, b)$ such that $\gamma' \setminus \{(0, 0, b)\} \subset QK^\circ$ and g approaches a limit along γ' . Let

$$\gamma = \{(x, y) \in R^{2*}, (x, y, 2) \in \gamma' \text{ for some } 2\}.$$

Then γ is a (not necessarily simple) arc with one endpoint at $(0, 0)$ and $\gamma \setminus \{(0, 0)\} \subset \mathbb{C}Q^\circ$. I assert that $g(x, y) \rightarrow 2$ approaches 0 along γ' .

Assume this is false. Then there exists $c > 0$ and there exists a sequence of points $\{(x_n, y_n, z_n)\}_{n=1}^\infty$ in $\gamma' \setminus \{(0, 0, b)\}$ such that

$$(x_n, y_n, z_n) \rightarrow (0, 0, b) \text{ as } n \rightarrow \infty$$

and $|g(x_n, y_n) - z_n|$ for all n . Let $\delta > 0$ be chosen so that whenever $(w, v, w) \in \mathcal{L}y'$, $(x, y, z) \in \mathcal{E}y'$, and $v, y \rightarrow b$, then $|w - z| < \delta$. Let N be chosen so that $n \geq N$ implies $y_n < \min\{3\epsilon/32, 33/4, 3/4\}$.

For the present, let n be a fixed integer greater than N . Set $a = y_n/3$. There exists an arc γ^* contained in

$$\gamma \setminus C \setminus \{(x, y) \in R^{2*}, |(x, y) - (x_n, y_n)| \leq d(a)\}$$

joining (x_n, y_n) to a point on the circle of radius $d(a)$ about (x_n, y_n) . Clearly (diameter γ^*) (a) , so (diameter $k(y^*)$) ≤ 2 . Choose points

(x'', y'') , (x', y') in γ^* with $|k(x', y') - k(x'', y'')| \leq 2$. Choose z_n, z'_n so that (x'_n, y'_n, z_n) and (x'', y'', z'_n) are in γ' . It is easy to check that $|x'_n| \leq 8$ and $|x''| \leq 8$, so

$$(2) \quad |z_n - z'_n| \leq \delta \text{ and } |z'_n - z_n| \leq \delta.$$

Moreover, since $|(x'_n, y'_n) - (x_n, y_n)| \leq d(a) \leq \delta d(a)$, we have

$ g(x_n, y_n) - g(x'_n, y'_n) $	$< \delta$	ϵ ;
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and similarly

$ g(x'', y'') - g(x_n, y_n) $	$\leq \delta$
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Combining these inequalities with (1) and (2), we get

$$\begin{aligned} & |f(x_n, y_n, z_n) - f(x'_n, y'_n, z'_n)| > \\ & |g(x_n, y_n, z_n) - g(x'_n, y'_n, z'_n)| \geq 2\delta - \delta. \end{aligned}$$

But $y'_n, y_n \rightarrow y_n/3$, so $(x'_n, y'_n, z'_n) \rightarrow (0, 0, b)$ and $(x'', y'', z'_n) \rightarrow (0, 0, b)$ as $n \rightarrow \infty$; hence f cannot approach a limit along γ' , which is a contradiction. We conclude that $g(x, y) \rightarrow 2$ as $(x, y, 2) \rightarrow (0, 0, b)$ along γ' .

It follows immediately that $g(x, y) \rightarrow b$ along γ , so $b \in E$. We have now shown that

$$\text{rnz} = \{(o,0,z): z \in \mathcal{L}\}.$$

Thus $TC \setminus L$ is not a Borel set. Hence T is not a Borel set; for if it were, then $TPiL$ would also be a Borel set.

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Ted's Work from his Parents Home in Illinois

11. Problem 786

January, 1971

<https://doi.org/10.2307%2F2688865>

By T. J. Kaczynski, Lombard, Illinois.

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

12. A Match Stick Problem

November–December 1971

<https://doi.org/10.2307%2F2688646>

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Problem 786. [January, 1971] Proposed by T. J. Kaczynski, Lombard, Illinois.

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

I. Solution by Richard A. Gibbs, Hiram Scott College, Nebraska.

A necessary and sufficient condition that a solution exist is that n be even.

Sufficiency is easy. If $n = 2k$, consider the cardboard as consisting of k^2 2×2 squares. Simply place a match on each of the four segments adjacent to the center point of each 2×2 square.

For necessity, assume a solution exists for an $n \times n$ sheet of cardboard. To each unit square correspond the point at its center. Connect two points if their corresponding squares share a match. By the hypotheses, every point will be joined to exactly two others. Therefore, according to a basic result of Graph Theory, the resulting graph will be a collection of disjoint cycles. Each cycle will enclose a polygonal region whose sides are either horizontal or vertical line segments. Consequently, since the length of each segment is an integer, the area of each polygonal region will be an integer. By Pick's theorem (a beautiful result familiar to anyone who has played with a geo-board) the area of the i th polygonal region is

$$A = \sum P_i + l_i - 1$$

where there are P_i points on the perimeter and l_i points in the interior of the 2th polygonal region. Since each area is an integer, each P_i is even. As each point is on exactly one perimeter, the sum of the P_i is the total number of points, $n/2$. Hence n is even.

II. Solution by Richard L. Breisch, Pennsylvania State University.

A generalization of the stated problem will be demonstrated. Let the cardboard be an $m \times n$ rectangle. The problem of covering the cardboard in the stated manner has a solution if and only if m and n are not both odd.

An alternative representation of the problem will be used to demonstrate this. Consider the $m \times n$ array of the center points of the little squares. If two edge-adjacent squares have a match on their mutual edge, connect the centers of these squares with a line segment. Since each little square has exactly two of its sides covered by matches, in the alternative representation, there are exactly two line segments from each point in the array. Hence each connected set of line segments forms a polygon, and the $m \times n$ array is covered by a collection of polygons. Each polygon must have an even number of horizontal segments and an even number of vertical segments. Since there are $m \times n$ segments, m and n cannot both be odd integers.

Suppose m is even. Then the $m \times n$ array can be covered with $m/2$ rectangular polygons each of which has dimensions 1 segment by n segments. The arrangement of matches in the original representation is easily derived from this representation.

Also solved by Dan Bean, Dave Harris and E. F. Schmeichel (Jointly), College of Wooster, Ohio; Thomas A. Brown, Santa Monica, California; Melvin H. Davis, New York University; Roger Engle and Necdet Ucoluk (jointly), Clarion State College Pennsylvania; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Heiko Harborth, Braunschweig, Germany; Herbert R. Leifer, Pittsburgh, Pennsylvania; Joseph V. Michalowicz, Catholic University of America; George A. Novacky, Jr., University of Pittsburgh; J. W. Pfaendtner, University of Michigan; Sally Ringland, Shippensburg, Pennsylvania; Rina Rubinfeld, New York City Community College; E. P. Starke, Plainfield, New Jersey; and the proposer.

Ted's Work as a Montana Hermit

Never published new ground?

Ted briefly went back to playing around with pure math equations in his cabin in Montana. He even submitted one paper to a journal in 1976 called Four-Digit Numbers that Reverse Their Digits When Multiplied which he claimed proved an earlier posited solution:¹

... Some time ago - (Last Nov. or Dec.) I submitted a mathematical paper for publication, and I am rather ashamed of this. Not because of any idea that the paper will advance technical progress — I feel confident that it will never have any practical applications, direct or indirect — but because it represents, to a certain degree, a personal surrender to one of the escape mechanisms which keep people distracted so that they can forget the purposelessness, subordination, and indignity of life in a technological society ...

Here's how Ted explained the paper in relation to his other work:²

(Ca) FL #80, letter from me to my parents, Spring, 1964, p. 1: "It's a good thing I didn't follow Piranian's suggestions about how to attack the problem, or I never would have solved it!"

Piranian urged me to prove (a) that every continuous function in the disk admits a family of disjoint arcs, and to deduce from this (b) that every boundary function for a continuous function can be made into a function of the first Baire class by changing its values on at most a countable set. (The terminology is explained in F. Bagemihl and G. Piranian, "Boundary Functions for Functions Defined in a Disk," *Michigan Mathematical Journal*, 8 (1961), pp. 201–207.)

I maintained that it would be much easier to prove (b) by examining inverse-image sets, and I even suggested that (b) might then be used to prove (a). And that's how it turned out. I did prove (b) within three months or so by using inverse-image sets. The proof of (a) was vastly more difficult. I didn't succeed in proving (a) until two decades later, and I had to use (b) in order to do it. The proof of (a) has not been published.

¹ A Review and Compilation of the Writings of Ted Kaczynski

² Conversations with the producer of this great documentary on Ted's math career: Ted Kaczynski's

His perspective of intellectual tension with Piranian is interesting and I maybe sensed it reading their papers. Piranian and collaborators were in the complex analysis space, it seemed like they were developing language for talking about boundary functions from their work on cluster sets, and all of it seemed geared toward complex analysis applications. Ted's papers treated boundary functions like its own distinct discipline and I don't think he mentions cluster sets once. So it's funny that George seemed to urge him to prove this statement (a) about "functions admitting disjoint arcs", a concept originating from cluster sets, while Ted proved this statement (b) first which is primarily a boundary function concept:

Statement (b) that continuous half plane functions admit honorary Baire class 2 functions (functions differing from baire class 1 on at most a countable subset) shows up in his thesis and it seems like he proved a version of that fairly early in 1964 based on that letter.

As far as statement (a) I was not too far in the weeds of cluster sets but tbh it sounds like something that should have been proved already. Wouldn't take my word for it over his though.

Finally, here's a glimpse into Ted's headspace when writing it, from a journal entry at the time:⁴

Ever since seeing how the Trout Creek area has been ruined I feel so much grief whenever I am sitting quietly, or when I am walking slowly through the woods just looking and listening, that I have to keep occupied almost all the time in order to escape this grief. That was my favorite spot. Whoever has read my notes knows very well what the other causes have been. Where can I go not to enjoy in peace nature and the wilderness life? — which are the best things I have ever known. Even in the officially designated "wilderness" there must be the continued noise of airplanes, especially the jets, since I know that planes are permitted to fly over the Bob Marshal and Scapegoat wildernesses. Are there fewer planes there than here. Maybe, maybe. Perhaps one of these days I'll go and find out. But so many times I've gone looking for a place where I can escape completely from industrial society, and always . . . [three dots in the original] well, I'm very discouraged. So, I've been playing around with mathematics a good deal lately. It's a rather contemptible game, but while I'm involved in it, it enables me to escape from my grief.

PhD Thesis

³ Truth versus Lies (Original Draft)

⁴ Journal #1 of 4 from Series VII (1984-1986)

Would the paper he wrote in his cabin get published today? Or would all the formulas and explanations have already been well covered by other papers?

SPV Laboratories:⁵

Can't speak to whether or not his stuff would get published today... if my paper gets published then that would be in his favor. The things complex analysts were interested in during the mid 20th century are not so much in vogue anymore, in fact it seems like they fell out of style quite fast based on what I've seen in the literature. So most things Ted would have been working on post-Berkeley would have been non-trivial but also probably not discovered sooner by anyone due to a lack of interest. Not implying anything negative about his work, just that certain inquiries in math come and go.

⁵ Conversations with the producer of this great documentary on Ted's math career: Ted Kaczynski's PhD Thesis

13. Four-Digit Numbers that Reverse Their Digits When Multiplied

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FOUR-DIGIT NUMBERS THAT REVERSE THEIR DIGITS WHEN MULTIPLIED

T. J. KACZYNSKI

If $n \geq 2$ is an integer and a_0, \dots, a_{n-1} are integers satisfying $0 \leq a_i < n$ for $i = 0, 1, \dots, n-1$,

then we let $(a_0, \dots, a_{n-1})_n$ denote the number

$\sum_{i=0}^{n-1} a_i n^i$. Whenever we write a symbol of the form $(a_0, \dots, a_{n-1})_n$ it is to be understood that

$0 \leq a_i < n$ for $i = 0, 1, \dots, n-1$ so that $(a_0, \dots, a_{n-1})_n$ are the digits of the number $(a_0, \dots, a_{n-1})_n$ in base n notation.

If k is an integer and $1 \leq k < n$, we say that $(a_0, \dots, a_{n-1})_n$ is *reversible for n, k* if and only if $a_0 \neq 0$ and $(a_0, \dots, a_{n-1})_n = k(a_{n-1}, a_{n-2}, \dots, a_0)_n$. Reversible numbers have been studied in¹, [2],². The purpose of this paper is to construct a rather involved family of 4-digit reversible numbers that illustrates the complexity of the reversible number problem. We use the abbreviation RN for "reversible number".

Sutcliffe [3] showed that there exists a 4-digit RN for any base $n \geq 3$. Let d be any divisor of n

(possibly n itself) with $d \geq 3$, and set $t = n/d$ and $k = d-1$. Then

$$k(t, t-1, n-t-1, n-t)_n = (n-t, n-t-1, t-1, t)_n.$$

(This family of

Let us refer to a RN of this type as a *Sutcliffe RN*. Note that the Sutcliffe reversible number $(t, t-1, n-t-1, n-t)_n$ is equal to $(n+1)(t-1, n-1, n-t)_n$.

At least two other types of 4-digit RNs may exist for certain values of n .

If $(a, b, c)_Q$ is a J -digit RN for n, k , and if $a+b \leq n-1$ and $b+c < n-1$, then $(n+1)(a, b, c)_n$ is a 4-digit RN for n, k . (For instance, $4X(2, 5, 9) =$

¹ F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci. U. S. A.* 4 (1955) 379-382.

² S. Banach, Über analytisch darstellbare Operationen in abstrakten Räumen, *Fund. Math.*, 17

$(9,5 \gg 2)_{17}$; multiplying by 18 yields $4X(2,7,14,9)_{17} \rightarrow (9,14,7,2) \bullet$

If $(a,b,c)_n$ is any solution of the system of conditions

$$k(a,b,c)_n = (c-1, b+1, a)_Q,$$

(1)

$$a+b \leq n-2, b+c \leq n, a \geq 0,$$

then $(n+1)(a,b,c)_Q$ is a 4-digit RN for n, k , as can

be verified by computation. We note that $4 \leq 7 \leq 10$

from a solution of (1) can never be a Sutcliffe RN for n, k , because if $t = n/(k+1)$ then $(t-1, n-1, n-t)_Q$ cannot satisfy (1).

One family of solutions of (1) can be obtained by taking any integers $u \geq 1$ and $k \geq 3$ and setting $n = u(k^2-1)+k$, $a = (k-1)u$, $b = u(k+1) + 1(k-2)$, $c = (uk+1)(k-1)$. Observe that the corresponding 4-digit RN is $(n+1)(a,b,c)_n = (k-1, k-3, k-1)_n(u, uk+1)_n$, and that $(u, uk+1)_n$ is a 2-digit RN for n, k .

Sutcliffe [J] showed that there exists a 2-digit RN in base n notation if and only if $n+1$ is not prime. It was shown in [1] that there exists a 3-digit RN for n if and only if $n+1$ is not prime. This directs our attention to 4-digit RNs in the case where $n+1$ is prime.

Does (1) ever have a solution when $n+1$ is prime? The answer is yes. With $n+1 = 59$ we have $19X(2,41,52)53 = (51,42,2)^{\wedge}$, which yields $19 \cdot (2,44,35,52)^{\wedge} = (52,35,44,2)53$.

Do there exist infinitely many such examples? The answer is again yes.- Let s be any nonnegative integer, take $k \leq 19$, $n = 5k+360s$, $a = 2+17a$, $b = 41+260s$, $c = 52+323s$, and we have a solution of (1). By Dirichlet's Theorem, there are infinitely many positive integers s for which $n+1 \leq 59+360s$ is prime.

However, all these solutions are in a sense

isomorphic; we do not regard them as essentially different. What we really want to show is this:

There exist infinitely many positive integers k having the property that there exist integers n, a, b, c for which $n+1$ is prime and the system of conditions (1) is satisfied.

This is our main result. To prove it, set

$$f(x) = 41067x^2 - 1067x - 9, g(x) = 10179x^2 - 222x - 1.$$

The discriminant of $g(x)$ is $8 \cdot 10^8 \gg 2^{1071}$, not a square, so $g(x)$ has no linear factor with rational coefficients. Therefore $f(x)$ and $g(x)$ have no nonconstant common factor with rational coefficients. Consequently there exist polynomials $p(x)$ and $q(x)$, with rational coefficients, such that $p(x)f(x) = q(x)g(x)$. Let $d > 0$ be the product of the denominators of all the fractions that appear as coefficients of $p(x)$ and $q(x)$, and let $P(x) = dp(x)$ and $Q(x) = dq(x)$. Then $P(x)$ and $Q(x)$ have integer coefficients and $P(x)f(x) = Q(x)g(x) = d$.

Let k be any number of the form $k \leq 117y-2$, where y is a positive integer. Let $D = yd$ and let v be the greatest common divisor of $f(D)$ and $g(D)$. Then v divides D and

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$yP(D)f(D) \not\equiv yQ(D)g(D)$. Since v divides $g(D)$ it follows that v divides 1. Thus $f(D)$ and $g(D)$ are relatively prime.

By Dirichlet's Theorem, we can choose a positive integer t for which $f(D)t + g(D)$ is prime. Set

$$\begin{aligned} n &= f(D)t + g(D) - 1 = (4iO67D^2 - UO4D^*9)t + 1O179D^2 - 222D, \\ u &= 1JD, r = 2(u-1), m = 117Dt - t^*29D = (9u-1)t + 29D, \\ U &= 3u-1, R = 3r+1 \text{ a } 6u-5 = 78D-5, M = 9m^*1, \\ w &= 9rm^*Jm^*r. \end{aligned}$$

We compute

$$k = 117D-2 \text{ a } 9u-2 = 3U+1, n \text{ a } MU^*1, MR = 3w^*1.$$

Modulo $9u-1$ we have the following congruences:

$$\begin{aligned} nR+w &\text{ a } (MU^*1)(6u-5)^*9rm^*3m+r \\ &\ll (27mu+3u-9m)(6u-5) \not\equiv 18mu-15m^*2u-2 \\ &= (3 \gg 3u-9m)(6u-5) 2ia-15m \gg 2u-2 \\ \text{a } 18u^2-13u-36mu+17m-2 &= -2u^*13m-3 \\ &= -26l \gg 377D-3 = 351D-3 \ll 3(117D-1) = 3(9u-1) \\ &= 0 \pmod{9u-1}. \end{aligned}$$

Thus $nR+w$ is divisible by $9u-1$. • Choose an integer c so that $(k+1)c = (9u-1)c \text{ a } nR+w$. Set $S = knR - (k^2-1)c-1$. Because $(n+1)R = (MU+2)R - 1 \pmod{3}$, we see that $k-1 = 3U$ divides $MU[(n+1)R-1]$. Thus

$$\begin{aligned} S_{n-R+1} &= (kn^2-1)R - (k^2-1)nc - (n-1) \\ E(n^2-1)R - (n-1) &= MU[(n+1)R-1] \equiv 0 \pmod{k-1}. \end{aligned}$$

Choose an integer b so that $(k-1)b = S_{n-R+1}$. Set $a = kc - Rn$. We then have

$$\begin{aligned} (2) \quad kc &= Rn+a \\ (3) \quad kb+R &= S_{n+b+1} \\ (4) \quad ka+S &= c-1 \end{aligned}$$

We must show that certain inequalities are satisfied. Clearly $2k < n, c > 2, 2R < k-1$. Thus $(k^2-1)c = 3U(k+1)c = 3U(nR+w) < 3UnR+UMR < 3UnR+nR \leq knR < kn(k-1) < (k^2-1)n$. So $2 < c < n$.

Observe that $R-1+U < 3U \leq 2(R-1)^*U$. Adding $3U(k^*1)c = 3V(nR+w)$ to this inequality gives

$$\begin{aligned} &< J \quad 3U(nR+w)4-R-1+U < 3U(k+1)c + 3U(nR+w)+2(R-1)+U, \quad (k-1)nR^*R- \\ &1+MRU < (k^2-1)c+k-1 -4 \quad (k-1)nR^*2R-2+MRU, \quad (k-1)nR > nR-1 < (k^2-1)c^*k-1 \\ &(k-1)nR+nR+R-2, \end{aligned}$$

$$1 < (k^2-1)c - knR + k+1 < R,$$

$$k-R < S < k-1$$

Thus $2 < S < k-1$ (from which we see that $b > 0$) and

$$(5) \quad S+R > k+1.$$

Also, $(k-1)b = S_{n-R+1} \leq S_{(k-2)n}$ so that $b < T < \frac{n}{k-2} = n-1$, and $b+1 < n$.

p

Note that $(k+1) \mid n$, so that $(k+1)c = nR+w > 2(k+1)$ and $c-1 > k > S$. • Thus $ka = c-1-S > 0$, so that $a > 0$.

From (J) and (4), we find $k(a+b)+R+S = S_{n+b+c} < (S+2)n \sim kn$ • Therefore $a+b < n$. Suppose $a+b = n-1$ • Then from (4) and the definition of b we have $(k-1)(n-1) = (k-1)(a+b) = S(n-1)+c-a-R$. Consequently $n-1$ divides $c-a-R$. But $c > ka$ by (4), so $n-1 > c-a-R > (k-1)a-R > 0$. This contradiction shows that $a+b \leq n-2$.

From (3) and (5) we see that $(k-1)(b+c) \leq S_{n-R+1}+(k+1)c-2c = (S+R)n+w+1-R-2c >$

$(k-1)n+w+1-R$. But $3R \leq 3w+1$, so that $R \leq w+1$. Therefore $b+c \geq n$ •

Equations (2), (3)» (4), together with the inequalities we have just proved, show that $(a,b,c)_n$ satisfies (1). ©

In the foregoing argument there is no need to restrict ourselves to the case where $n+1$ is prime, so the construction also yields many 4-digit RNs for composite values of $n+1$.

We hope to publish at a later date a more general treatment of reversible numbers, in which we shall prove (among other things) that if $n+1$ is prime, then every 4-digit RN for n is either a Sutcliffe RN, or of the form $(n+DfajbjC)$, where $(a,b,c)_n$ is a solution of (1).

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14. Handwritten Math equations and procedures

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Ted's Work from Prison

He would sometimes write hard questions for the teachers of kids who would write to him in prison:¹

Now I'm going to play a really nasty trick on your teacher. I'm going to give you a problem to give her, and if she doesn't get it right, you be sure to give her an F.

SPV Laboratories:²

One of these letters had some conjectures based on a question he posed in his thesis. He claims to have had proofs for three of these. I am able to prove two; I plan on publishing my results soon.

Finally, here is a link to that paper:

An uncountable union of line segments with null two-dimensional measure by Parker Kuklinski

¹ Ted Kaczynski, Math Tutor

² Conversations with the producer of this great documentary on Ted's math career: Ted Kaczynski's PhD Thesis

The Ted K Archive

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The Mathematical Work of Ted Kaczynski
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