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FOUR-DIGIT NUMBERS THAT REVERSE THEIR DIGITS WHEN MULTIPLIED

T. J. KACZYNSKI

If $n \geq 2$ is an integer and $a_0, a_1, \dots, a_h$ are integers satisfying $0 \leq a_i < n$ for $i = 0,1,\dots,h$, then we let $(a_h, \dots, a_1, a_0)_n$ denote the number $\sum_{j=0}^{h} a_j n^j$. Whenever we write a symbol of the form $(a_h, \dots, a_1, a_0)_n$, it is to be understood that $0 \leq a_i < n$ for $i = 0,1,\dots,h$, so that $a_h, \dots, a_1, a_0$ are the digits of the number $(a_h, \dots, a_1, a_0)_n$ in base n notation.

If k is an integer and $1 < k < n$, we say that $(a_h, \dots, a_1, a_0)_n$ is <u>reversible for</u> n, k if and only if $a_h \neq 0$ and $k(a_h, \dots, a_1, a_0)_n = (a_0, a_1, \dots, a_h)_n$. Reversible numbers have been studied in [1], [2], [3]. The purpose of this paper is to construct a rather involved family of 4-digit reversible numbers that illustrates the complexity of the reversible number problem. We use the abbreviation RN for "reversible number".

Sutcliffe [3] showed that there exists a 4-digit RN for any base $n \geq 3$. Let d be any divisor of n

(possibly n itself) with $d \geq 3$, and set $t = n/d$ and $k = d-1$. Then

$$k(t,\ t-1,\ n-t-1,\ n-t)_n = (n-t,\ n-t-1,\ t-1,\ t)_n .$$

(This family of RNs was rediscovered [redacted] in [2].) Let us refer to a RN of this type as a Sutcliffe RN. Note that the Sutcliffe reversible number $(t,\ t-1,\ n-t-1,\ n-t)_n$ is equal to $(n+1)(t-1,\ n-1,\ n-t)_n$.

At least two other types of 4-digit RNs may exist for certain values of n.

If $(a,b,c)_n$ is a 3-digit RN for n, k, and if $a+b \leq n-1$ and $b+c \leq n-1$, then $(n+1)(a,b,c)_n$ is a 4-digit RN for n, k. (For instance, $4 \times (2,5,9)_{17} = (9,5,2)_{17}$; multiplying by 18 yields $4 \times (2,7,14,9)_{17} = (9,14,7,2)_{17}$.)

If $(a,b,c)_n$ is any solution of the system of conditions

$$(1) \qquad \begin{aligned} &k(a,b,c)_n = (c-1,b+1,a)_n , \\ &a+b \leq n-2 ,\quad b+c \geq n ,\quad a \neq 0 , \end{aligned}$$

then $(n+1)(a,b,c)_n$ is a 4-digit RN for n, k, as can be verified by computation. We note that [redacted] a RN der from a solution of (1) can never be a Sutcliffe RN for n, k , because if $t = n/(k+1)$ then $(t-1, n-1, n-t)_n$ cannot satisfy (1).

One family of solutions of (1) can be obtained by taking any integers $u \geq 1$ and $k \geq 3$ and setting $n = u(k^2-1)+k$, $a = (k-1)u$, $b = (u(k+1)+1)(k-2)$, $c = (uk+1)(k-1)$. Observe that the corresponding 4-digit RN is $(n+1)(a,b,c)_n = (k-1,k-3,k-1)_n(u,uk+1)_n$, and that $(u,uk+1)_n$ is a 2-digit RN for n, k .

Sutcliffe [3] showed that there exists a 2-digit RN in base n notation if and only if n+1 is not prime. It was shown in [1] that there exists a 3-digit RN for n if and only if n+1 is not prime. This directs our attention to 4-digit RNs in the case where n+1 is prime.

Does (1) ever have a solution when n+1 is prime? The answer is yes. With n+1 = 59 we have $19 \times (2,41,52)_{58} = (51,42,2)_{58}$, which yields $19 \times (2,44,35,52)_{58} = (52,35,44,2)_{58}$.

Do there exist infinitely many such examples? The answer is again yes. Let s be any nonnegative integer, take $k = 19$, $n = 58+360s$, $a = 2+17s$, $b = 41+260s$, $c = 52+323s$, and we have a solution of (1). By Dirichlet's Theorem, there are infinitely many positive integers s for which $n+1 = 59+360s$ is prime.

However, all these solutions are in a sense isomorphic; we do not regard them as essentially different. What we really want to show is this:

There exist infinitely many positive integers k *having the property that there exist integers* n, a, b, c *for which* $n+1$ *is prime and the system of conditions* (1) *is satisfied*.

This is our main result. To prove it, set

$$f(x) = 41067x^2 - 1404x + 9$$

$$g(x) = 10179x^2 - 222x + 1 .$$

The discriminant of $g(x)$ is $8568 = 2^3 \cdot 1071$, not a square, so $g(x)$ has no linear factor with rational coefficients. Therefore $f(x)$ and $g(x)$ have no nonconstant common factor with rational coefficients. Consequently there exist polynomials $p(x)$ and $q(x)$, with rational coefficients, such that $p(x)f(x) + q(x)g(x) = 1$. Let $d > 0$ be the product of the denominators of all the fractions that appear as coefficients of $p(x)$ and $q(x)$, and let $P(x) = dp(x)$ and $Q(x) = dq(x)$. Then $P(x)$ and $Q(x)$ have integer coefficients and $P(x)f(x) + Q(x)g(x) = d$.

Let k be any number of the form $k = 117yd-2$, where y is a positive integer. Let $D = yd$ and let

v be the greatest common divisor of $f(D)$ and $g(D)$. Then v divides $D = yP(D)f(D) + yQ(D)g(D)$. Since v divides $g(D)$ it follows that v divides 1. Thus $f(D)$ and $g(D)$ are relatively prime.

By Dirichlet's Theorem, we can choose a positive integer t for which $f(D)t + g(D)$ is prime. Set

$$n = f(D)t + g(D) - 1 = (41067D^2-1404D+9)t+10179D^2-222D ,$$

$$u = 13D , \quad r = 2(u-1) , \quad m = 117Dt-t+29D = (9u-1)t+29D ,$$

$$U = 3u-1 , \quad R = 3r+1 = 6u-5 = 78D-5 , \quad M = 9m+1 ,$$

$$w = 9rm+3m+r .$$

We compute

$$k = 117D-2 = 9u-2 = 3U+1 , \quad n = MU+1 , \quad MR = 3w+1 .$$

Modulo $9u-1$ we have the following congruences:

$$\begin{aligned}
nR+w &= (MU+1)(6u-5)+9rm+3m+r \\
&= (27mu+3u-9m)(6u-5)+18mu-15m+2u-2 \\
&\equiv (3m+3u-9m)(6u-5)+2m-15m+2u-2 \\
&= 18u^2-13u-36mu+17m-2 \equiv -2u+13m-3 \\
&\equiv -26D+377D-3 = 351D-3 = 3(117D-1) = 3(9u-1) \\
&\equiv 0 \pmod{9u-1} .
\end{aligned}$$

Thus $nR+w$ is divisible by $9u-1$. Choose an integer c

so that $(k+1)c = (9u-1)c = nR+w$. Set $S = knR-(k^2-1)c-1$. Because $(n+1)R = (MU+2)R \equiv 1 \pmod 3$, we see that $k-1 = 3U$ divides $MU[(n+1)R-1]$. Thus

$$Sn-R+1 = (kn^2-1)R-(k^2-1)nc-(n-1)$$
$$\equiv (n^2-1)R-(n-1) = MU[(n+1)R-1] \equiv 0 \pmod{k-1} .$$

Choose an integer b so that $(k-1)b = Sn-R+1$. Set $a = kc-Rn$. We then have

(2) $$kc = Rn+a$$

(3) $$kb+R = Sn+b+1$$

(4) $$ka+S = c-1 .$$

We must show that certain inequalities are satisfied. Clearly $2 < k < n$, $c > 2$, $2 < R < k-1$. Thus $(k^2-1)c = 3U(k+1)c = 3U(nR+w) < 3UnR+UMR < 3UnR+nR = knR < kn(k-1) < (k^2-1)n$. So $2 < c < n$.

Observe that $R-1+U < 3U < 2(R-1)+U$. Adding $3U(k+1)c = 3U(nR+w)$ to this inequality gives

$$3U(nR+w)+R-1+U < 3U(k+1)c+\text{[illegible]} < 3U(nR+w)+2(R-1)+U ,$$
$$(k-1)nR+R-1+MRU < (k^2-1)c+k-1 < (k-1)nR+2R-2+MRU ,$$
$$(k-1)nR+nR-1 < (k^2-1)c+k-1 < (k-1)nR+nR+R-2 ,$$
$$1 < (k^2-1)c-knR+k+1 < R ,$$
$$k-R < S < k-1 .$$

Thus $2 < S < k-1$ (from which we see that $b > 0$) and

(5) $$S+R \geq k+1 .$$

Also, $(k-1)b = Sn-R+1 < Sn \leq (k-2)n$, so that $b < \frac{k-2}{k-1}n < \frac{n-1}{n}n = n-1$, and $b+1 < n$.

Note that $(k+1)^2 < n$, so that $(k+1)c = nR+w > (k+1)^2$ and $c-1 > k > S$. Thus $ka = c-1-S > 0$, so that $a > 0$.

From (3) and (4), we find $k(a+b)+R+S = Sn+b+c < (S+2)n \leq kn$. Therefore $a+b < n$. Suppose $a+b = n-1$. Then from (4) and the definition of b we have $(k-1)(n-1) = (k-1)(a+b) = S(n-1)+c-a-R$. Consequently $n-1$ divides $c-a-R$. But $c > ka$ by (4), so $n-1 > c-a-R > (k-1)a-R > 0$. This contradiction shows that $a+b \leq n-2$.

From (3) and (5) we see that $(k-1)(b+c) = Sn-R+1+(k+1)c-2c = (S+R)n+w+1-R-2c \geq (k+1)n+w+1-R-2c > (k-1)n+w+1-R$. But $3R < MR = 3w+1$, so that $R < w+1$. Therefore $b+c > n$.

Equations (2), (3), (4), together with the inequalities we have just proved, show that $(a,b,c)_n$ satisfies (1). ∎

In the foregoing argument there is no need to restrict ourselves to the case where $n+1$ is prime, so the construction also yields many 4-digit RNs for composite values of $n+1$.

We hope to publish at a later date a more general treatment of reversible numbers, in which we shall prove (among other things) that if $n+1$ is prime, then every 4-digit RN for n is either a Sutcliffe RN, or of the form $(n+1)(a,b,c)_n$, where $(a,b,c)_n$ is a solution of (1).

REFERENCES

1. T. J. Kaczynski, Note on a problem of Alan Sutcliffe, Math. Mag. 41 (1968), 84-86.

2. L. F. Klosinski and D. C. Smolarski, On the reversing of digits, Math. Mag. 42 (1969), 208-210.

3. A. Sutcliffe, Integers that are multiplied when their digits are reversed, Math. Mag. 39 (1966), 282-287.