A Match Stick Problem

787. [January, 1971] *Proposed by T. J. Kaczynski, Lombard, Illinois.*

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

I. *Solution by Richard A. Gibbs, Hiram Scott College, Nebraska.*

A necessary and sufficient condition that a solution exist is that n be even.

Sufficiency is easy. If $n=2k$, consider the cardboard as consisting of k^2 2×2 squares. Simply place a match on each of the four segments adjacent to the center point of each 2×2 square.

For necessity, assume a solution exists for an $n\times n$ sheet of cardboard. To each unit square correspond the point at its center. Connect two points if their corresponding squares share a match. By the hypotheses, every point will be joined to exactly two others. Therefore, according to a basic result of Graph Theory, the resulting graph will be a collection of disjoint cycles. Each cycle will enclose a polygonal region whose sides are either horizontal or vertical line segments. Consequently, since the length of each segment is an integer, the area of each polygonal region will be an integer. By Pick's theorem (a beautiful result familiar to anyone who has played with a geo-board) the area of the ith polygonal region is

$$A = \tfrac{1}{2}P_i + I_i - 1$$

where there are P_i points on the perimeter and I_i points in the interior of the ith polygonal region. Since each area is an integer, each P_i is even. As each point is on exactly one perimeter, the sum of the P_i is the total number of points, n^2. Hence n is even.

II. *Solution by Richard L. Breisch, Pennsylvania State University.*

A generalization of the stated problem will be demonstrated. Let the cardboard be an $m\times n$ rectangle. The problem of covering the cardboard in the stated manner has a solution if and only if m and $n\geqq 2$, and m and n are not both odd.

An alternative representation of the problem will be used to demonstrate this. Consider the $m\times n$ array of the center points of the little squares. If two edge-adjacent squares have a match on their mutual edge, connect the centers of these squares with a line segment. Since each little square has exactly two of its sides covered by matches, in the alternative representation, there are exactly two line segments from each point in the array. Hence each connected set of line segments forms a polygon, and the $m\times n$ array is covered by a collection of polygons. Each polygon must have an even number of horizontal segments and an even number of vertical segments. Since there are $m\cdot n$ segments, m and n cannot both be odd integers.

Suppose m is even. Then the $m\times n$ array can be covered with $m/2$ rectangular polygons each of which has dimensions 1 segment by n segments. The arrangement of matches in the original representation is easily derived from this representation.

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