but should reveal scholarship appropriate to the level of the student. The judging in March will be based on a 15 minute oral presentation of the paper by the writer to a panel of judges.

Judges are urgently needed to ensure the success of the Fair. Mathematics teachers and professors interested in performing this service on March 5 or at the March 12 final round please contact Professor Barz.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before May 15, 1972.

PROBLEMS

810. *Proposed by Zalman Usiskin, University of Michigan.*

Solve the following cryptarithm (in base 10):

$$\begin{array}{rcccccc}
 & & & & S & I & X \\
\times & & & & T & W & O \\
\hline
 & & & _ & _ & _ & _ \\
 & & _ & _ & _ & _ & \\
 & _ & _ & _ & _ & & \\
\hline
 & T & W & E & L & V & E
\end{array}$$

811. *Proposed by James Bookey, Mount Senario College, Wisconsin.*

Find a domain bounded by a simple closed polygon such that for each two sides of the polygon there is an interior point from which these two sides are visible, but such that there is no interior point from which all sides are visible.

812. *Proposed by Donald P. Minassian, Butler University, Indiana.*

Two fully ordered groups G and H are order-isomorphic if there is an isomorphism f from G to H which preserves orderings: $a>b$ in G if and only if $f(a)>f(b)$ in H. Let R be the additive group of real numbers under the usual ordering. Show that no proper extension, and no proper subgroup of R is order-isomorphic to R: we assume that the extension and subgroup are ordered to preserve the ordering of R.

813. *Proposed by L. Carlitz and R. A. Scoville, Duke University.*

Show that any polynomial with real coefficients can be written as a difference of two real monotone increasing polynomials.

814. *Proposed by Marlow Sholander, Case Western Reserve University.*

For what values of a is the graph of a^x tangent to the graph of $\log_a x$?

815. *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Given two complex numbers z_1 and z_2 whose sum is z. Let the angle between the vectors $0z_1$ and $0z$ be designated by θ. Let $0z_3$ and z_1z_3 intersect at z_3 such that:

(a) angle $z_2 0 z_3=\theta=$ angle $0z_1z_3$,

(b) the vector $0z_3$ lies within the angle ϕ subtended by $0z_1$ and $0z_2$; $0<\phi<\pi$.

Prove that $z_3=(z_1^{-1}+z_2^{-1})^{-1}$.

816. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Show that no equilateral triangle which is either inscribed in or circumscribed about a noncircular ellipse can have its centroid coincide with the center of the ellipse.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q528. If $781\times 965=753, A65$, without multiplying find A.

[*Submitted by P. G. Pantelidakis*]

Q529. Show that $\int_0^1 [1-(1-t)^n]t^{-1}dt=1+1/2+\cdots+1/n$.

[*Submitted by Frank J. Papp*]

Q530. Suppose $a-1$ and $a+1$ are twin primes larger than 10. Prove that a^3-4a is divisible by 120.

[*Submitted by George E. Andrews*]

Q531. There are many closed subsets of a solid torus which, like the meridian sections, intercept each longitudinal circle in the torus exactly once. Are there any which also contain the points of a simple (closed) spiral on the surface?

[*Submitted by R. A. Struble*]

(Answers on pages 298–299.)

SOLUTIONS

Nested Simplexes

782. [January, 1971] *Proposed by Herta T. Freitag, Hollins, Virginia.*

1. Into an n-dimensional regular simplex of side a an n-dimensional hypersphere is inscribed. Inscribe another regular simplex into this hypersphere. Continue in this manner *ad infinitum*. Find:

a. the total contents of the entire set of simplexes.

b. the total contents of the entire set of hyperspheres.

2. Deal with the analogous problem concerning an n-dimensional hypercube.

Solution by Michael Goldberg, Washington, D.C.

(1) The radius r of an inscribed hypersphere of n-dimensional regular simplex of edge a is given by the equation

$$r = \{1/2n(n+1)\}^{1/2}a.$$

(See *Regular Polytopes*, H. S. M. Coxeter, in table of page 295).

The radius R of the circumsphere is nr. The edge a_1 of the simplex in the inscribed hypersphere is given by the equation

$$a_1 n\{1/2n(n+1)\}^{1/2} = r = \{1/2n(n+1)\}^{1/2}a.$$

Hence, $a_1 = a/n$. Similarly, $a_2 = a/n^2$, $a_3 = a/n^3$, etc.

The content C of a regular simplex of edge a is given by the equation

$$C = \frac{a^n}{n!}\left(\frac{n+1}{2^n}\right)^{1/2}.$$

Hence, the total contents of the entire set of simplexes is given by the equation

$$\frac{1}{n!}\left(\frac{n+1}{2^n}\right)^{1/2}\sum a_k^n = \frac{a^n}{n!}\left(\frac{n+1}{2^n}\right)^{1/2}\sum_{k=0}^{\infty} 1/n^k$$
$$= \frac{a^n}{n!}\left(\frac{n+1}{2^n}\right)^{1/2}\left(\frac{n}{n-1}\right).$$

The content S_n of a hypersphere of radius R in n dimensions is given by the equation $S_n = 2\pi^{n/2}R^n/\Gamma(n/2)$. (Coxeter, pp. 125–126.) Hence, the sum of the contents of the hyperspheres is given by

$$\frac{2\pi^{n/2}}{(n/2)}\{1/2n(n+1)\}^{n/2}a^n\sum_{k=0}^{\infty} 1/n^k = \frac{2\pi^{n/2}}{\Gamma(n/2)}\{1/2n(n-1)\}^{n/2}a^n\left(\frac{n}{n-1}\right).$$

(2) For hypercubes, the ratios of the lengths of the successive edges is $1/\sqrt{n}$. Hence, the sum of the contents of the hypercubes is

$$\sum_{k=0}^{\infty} 1/n^{k/2} = \sqrt{n}/(\sqrt{n}-1).$$

Similarly, the sum of the contents of the hyperspheres is

$$\frac{2\pi^{n/2}(a/2)^n}{\Gamma(n/2)}\left(\frac{\sqrt{n}}{\sqrt{n}-1}\right).$$

Also solved by Heiko Harborth, Braunschweig, Germany and the proposer.

Saalschützian Series

783. [January, 1971] *Proposed by H. L. Krall, University Park, Pennsylvania.*

1. If a_{ij} is the binomial coefficient $\binom{p}{q-i+j}$, evaluate d_n where

$$d_n = \begin{vmatrix} a_{11} & \cdots & a_{1^n} \\ \vdots & & \\ a_{n_1} & \cdots & a_{nn} \end{vmatrix}.$$

2. Show that $\sum_{k=0}^{n}\binom{n}{k}x_{\underline{k|}}y_{\underline{k|}}z_{\underline{n-k|}}(x+y+z-k)_{\underline{n-k|}}=(x+z)_{\underline{n|}}(y+z)_{\underline{n|}}$ where

$$x_{\underline{n|}} = x(x-1)\cdots(x-n+1).$$

Solution by H. W. Gould, West Virginia University.

By adding rows successively in the given determinant and using the recurrence relation $\binom{x}{i-1}+\binom{x}{i}=\binom{x+1}{i}$, it is easy to see that the given nth order determinant with general element $a_{ij}=\binom{p}{q-i+j}$ has the same value as the nth order determinant with general element $a_{ij}=\binom{p+i-1}{q+j-1}$. Recalling that $\binom{x}{i}=\frac{x}{i}\binom{x-1}{i-1}$ and using this repeatedly, we see that the determinant factors, and indeed, p is a factor of every element in the first row, $1/q$ is a factor of every element in the first column, $p+1$ is a factor of every element in the second row, $1/(q+1)$ is a factor of every element in the second column, etc. Then the ratio $(p-1)/(q-1)$ is seen to be a factor, $(p-2)/(q-2)$, etc., and carrying this factorization to its conclusion we obtain finally the desired formula for the value of the determinant:

$$D_n = \prod_{i=0}^{n-1}\frac{\binom{p+i}{q+1}}{\binom{p-q+i}{i}}.$$

The determinant seems first to have been evaluated by V. Zeipel in 1865. Details of his work and other valuable references may be found in the book *Die Determinanten*, by Ernesto Pascal, Teubner, Leipzig, 1900, pp. 133–134. The method above is that given by Pascal. Pascal gives other valuable references in the older literature dealing with determinants whose elements are binomial coefficients.

The formula to be proved may be stated in the equivalent form

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z-k}{n-k} = \binom{x+z}{n}\binom{y+z}{n}, \tag{1}$$

and the formula will be shown to be true for all real or complex x, y, z, and integers $n \geqq 0$. There are several ways to prove the formula using formulas for hypergeometric functions; however, our proof below will use just the Vandermonde convolution (addition theorem)

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}, \tag{2}$$

known to be true for all real or complex x and y, and some operations on finite series. We shall first prove (1) in the special case that x is a nonnegative integer and then show that the extension to arbitrary x follows at once. To be precise, (1) is a polynomial identity in x, y, z, and so if (1) can be established for more values of x than the degree of the polynomial in x, it is easy to see that the formula must then be true for all real x.

Now it is easy to see that

$$\binom{x}{k}\binom{z}{n-k}\binom{n}{k}^{-1} = \binom{x+z-n}{x-k}\binom{x+z}{n}\binom{x+z}{x}^{-1} \tag{3}$$

when x is a nonnegative integer. Using this, it is clear that (1) is equivalent to the formula

$$\sum_{k=0}^{n} \binom{y}{k}\binom{x+y+z-k}{n-k}\binom{x+z-n}{x-k} = \binom{x+z}{x}\binom{y+z}{n} \tag{4}$$

which we will prove valid for nonnegative integers x, n and real y, z. In fact

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{y}{k}\binom{x+y+z-k}{n-k}\binom{x+z-n}{x-k} \\
&\qquad = \sum_{k=0}^{n} \binom{y}{k}\binom{x+z-n}{x-k}\sum_{j=0}^{n-k}\binom{x+z}{j}\binom{y-k}{n-k-j}, \qquad \text{by (2)} \\
&\qquad = \sum_{j=0}^{n} \binom{x+z}{j}\sum_{k=0}^{n-j}\binom{y}{k}\binom{x+z-n}{x-k}\binom{y-k}{n-j-k}, \\
&\qquad = \sum_{j=0}^{n} \binom{x+z}{n-j}\sum_{k=0}^{j}\binom{y}{k}\binom{x+z-n}{x-k}\binom{y-k}{j-k}, \\
&\qquad = \sum_{j=0}^{n} \binom{x+z}{n-j}\binom{y}{j}\sum_{k=0}^{j}\binom{j}{k}\binom{x+z-n}{x-k},
\end{aligned}$$

$$= \sum_{j=0}^{n} \binom{x+z}{n-j}\binom{y}{j}\binom{x+z-n+j}{x}, \qquad \text{by (2)}$$

$$= \binom{x+z}{x} \sum_{j=0}^{n} \binom{y}{j}\binom{z}{n-j},$$

$$= \binom{x+z}{x}\binom{y+z}{n}, \qquad \text{by (2).}$$

The proof of (4) is then complete. Converting this back into the form (1) of a polynomial identity in x, y, z, the general result follows.

We remark finally that (1) may be cast in the equivalent form

$$(5) \qquad \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{y}{k}\binom{z}{n-k}}{\binom{x+y+z}{k}} = \frac{\binom{x+z}{n}\binom{y+z}{n}}{\binom{x+y+z}{n}}$$

which is also valid for all real or complex x, y, z. This form is closely related to a form of Saalschutz.

Also solved by L. Carlitz, Duke University; Eldon Hansen, Lockheed Research Laboratory, Palo Alto, California; and the proposer.

A Ring Property

784. [January, 1971] *Proposed by Robert S. Doran, Texas Christian University.*

Let A be a ring with identity 1 without divisors of zero, and suppose $f: A \to A$ is a ring antihomomorphism such that $f(f(a)) = a$. Show that a and $f(a)$ commute whenever $af(-a) = 1$.

I. *Solution by Eugen Peter Bauhoff, Frankfurt, West Germany.*

Suppose that $a \cdot f(a) = 1$ and put $f(a) = x$. Then $f(-a) = -x$ and

$$(1) \qquad a \cdot (-x) = 1.$$

We have to show that $a \cdot x = x \cdot a$.

Multiplying (1) with a from the right, we get

$$(2) \qquad a \cdot (-x) \cdot a = a.$$

Therefore

$$(3) \qquad a \cdot (-x) \cdot a - a = a((-x) \cdot a - 1) = 0.$$

Since A contains no divisors of zero and since $a \neq 0$ by (1), we get

$$(4) \qquad (-x) \cdot a = 1.$$

From (1) and (4) it follows immediately that $a \cdot x = x \cdot a$.

The condition $f(f(a)) = a$ and the antihomomorphism-property of f have not been used in the proof.

II. *Solution by Bob Prielipp, Wisconsin State University at Oshkosh.*

We shall establish the following stronger result: Let A be a ring with identity 1 without divisors of zero, and suppose $f:A\to A$ such that $f(x+y)=f(x)+f(y)$ for each $x, y\in A$. Then a and $f(b)$ commute whenever $af(-b)=1$.

By hypothesis $f(x+y)=f(x)+f(y)$. Thus $f(0)=0$. Hence $0=f(0)=f(b+(-b))=f(b)+f(-b)$ and $f(-b)=-f(b)$.

If $a=0$ then clearly $af(b)=f(b)a$. Thus in the remainder of our solution we shall assume that $a\neq 0$. By hypothesis $af(-b)=1$. Hence $a(f(-b)a)=(af(-b))a=1\cdot a=a\cdot 1$. Since A has no divisors of zero, $f(-b)a=1$. Thus $af(-b)=f(-b)a$ or $a(-f(b))=(-f(b))a$. Therefore $-(af(b))=-(f(b)a)$, from which it follows immediately that $af(b)=f(b)a$.

Also solved by Arthur R. Bolder, Brooklyn, New York; Edmund M. Clarke, Madison College; George Corliss, Michigan State University; William F. Fox, Moberly Junior College, Missouri; Stephen I. Gendler, Clarion State College, Pennsylvania; M. G. Greening, University of New South Wales, Australia; Greg Jennings, University of Puget Sound, Washington; Richard Kerns, Oak Lawn Illinois; Henry S. Lieberman, Waban, Massachusetts; David E. Manes, State University College, Oneonta, New York; W. Margolis, Colgate University; Joseph V. Michalowicz, Catholic University of America; C. Bruce Myers, Austin Peay State University, Tennessee; Thomas O'Loughlin, SUNY, Cortland, New York; Robert B. Reisel, Loyola University of Chicago; Rina Rubenfeld, New York City Community College, Brooklyn, New York; E. F. Schmeichel, College of Wooster, Ohio; Edward C. Waymire, Southern Illinois University; Albert White, St. Bonaventure University, New York; E. T. Wong, Oberlin College, Ohio; and the proposer, who resubmitted the problem with the weakened hypothesis.

Triangles in a Square

785. [January, 1971] *Proposed by Michael Goldberg, Washington, D.C.*

In Problem 745, this MAGAZINE, solution May, 1970, we were asked "Given any nine points in a unit square, show that among the triangles having vertices on the given points there exists at least one triangle whose area does not exceed 1/8." Closer bounds can be found.

a. In particular, find an arrangement of ten points such that the smallest triangle has an area of $(3\sqrt{17}-11)/32=0.0428$.

b. Find an arrangement of nine points in which the smallest triangle has an area greater than in a but less than 1/8.

Solution by the proposer.

(a) Eight points are placed on the edges of the square, two on each side. Each point is at a distance of $(5-\sqrt{17})/4$ from a corner. Each of the two remaining points is placed on a diagonal of the square at a distance of $\sqrt{2}(9-\sqrt{17})/16$ from a corner. Then the area of the smallest triangle is $(3\sqrt{17}-11)/32=0.0428$.

(b) Consider the arrangement of the nine points which are shown as circles points labelled A to I in the figure. Let the distances x, y, z be unknown. Then, if the areas K of the small triangles are equated, we can solve for x, y, z.

From

$$K(AGH) = K(AFE) = K(ACH) = xy$$
$$= K(EIC) = (1 - 2y - 2z + 2xz)/4$$

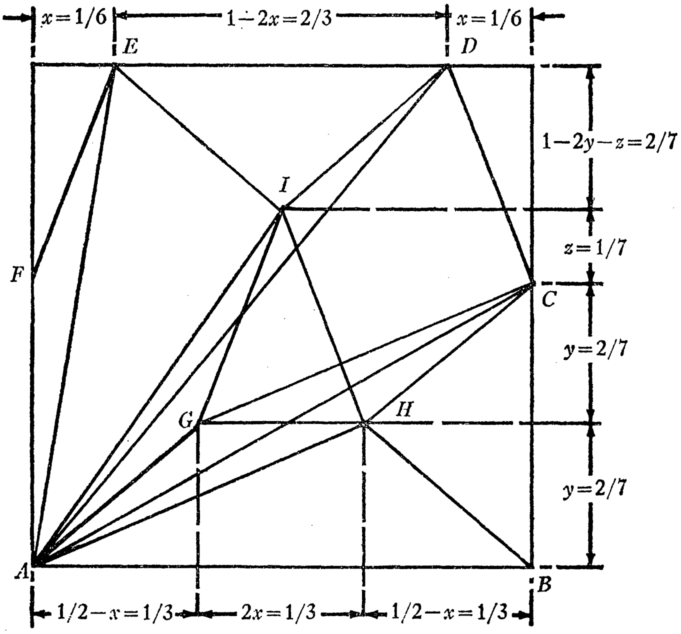


FIG. 1. Nine Points in square
Area of smallest triangle $=1/21=0.04762$.

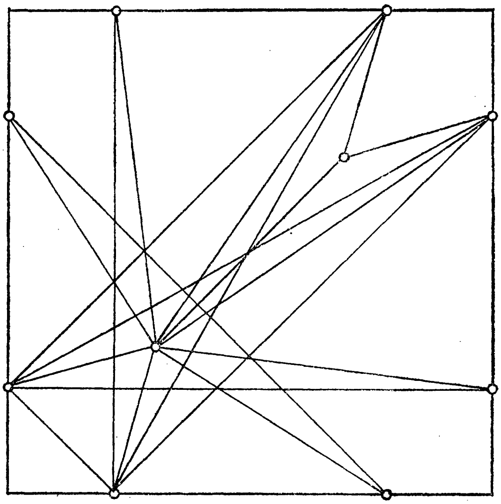


FIG. 2. Ten points in Square
Area of smallest triangle $=K=(3\sqrt{17}-11)/32=0.0428$.

$$= K(EIH) = (y + z - 2x + 2xy)/4$$
$$= K(AGI) = (y + z - 4xy - 2xz)/4,$$

we obtain $x=1/6$, $y=2/7$, $z=1/7$ and $K=1/21=0.04762$.

Other arrangements of the nine points were considered by the proposer, but none yielded a greater least triangle. A rigorous proof of the nonexistence of a better arrangement has not been developed.

A Square Number with Five Fours

786. [January, 1971] *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.*

Let N be an integer with n digits, in the decenary system, such that the $(n-1)$th, $(n-2)$th, $(n-3)$th, $(n-4)$th, and $(n-5)$th digits are all 4's. What is the smallest N that is a perfect square?

Solution by E. P. Starke, Plainfield, New Jersey.

If the $(n-1)$th digit is 4, the nth must be 1 or 4 or 9. We start with $(50a\pm 7)^2\equiv 49 \pmod{10^2}$ and find x so that

$$(50x \pm 7)^2 \equiv 449 \pmod{10^3}.$$

The solution of this congruence is routine and leads to

$$(500a \pm 107)^2 \equiv (500a \pm 143)^2 \equiv 449 \pmod{10^3}.$$

Adding a digit at a time we find similarly in succession

$$(5000a \pm 393)^2 \equiv (5000a \pm 857)^2 \equiv 4449 \pmod{10^4},$$
$$(50000a \pm 15393)^2 \equiv (50000a \pm 15857)^2 \equiv 44449 \pmod{10^5},$$
$$(500000a \pm 15857)^2 \equiv (500000a \pm 234707)^2 \equiv 444449 \pmod{10^6}.$$

Thus the smallest number whose square$= \cdots 444449$ is 15857.

Similar analysis produces

$$(500000a \pm 82021)^2 \equiv (500000a \pm 175771)^2 \equiv 444441 \pmod{10^6}$$

and the smallest number of this form is 82021.

Now we have $(500a\pm 38)^2=444 \pmod{10^3}$ but, for all a, the next digit (the $(n-4)$th) must be odd.

Note that $(324229)^2=105124444441$.

If we counted digits from the other end we could find the smallest: $120185^2 =14444434225$.

Also solved by Leon Bankoff, Los Angeles, California; Richard L. Breisch, Pennsylvania State University; Mannis Charosh, Brooklyn, New York; Heiko Harborth, Braunschweig, Germany; E. F. Schmeichel and Dave Harris (jointly), College of Wooster, Ohio; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Lewisburg, Pennsylvania; and the proposer. Five incorrect solutions were received.

A Match Stick Problem

787. [January, 1971] *Proposed by T. J. Kaczynski, Lombard, Illinois.*

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

I. *Solution by Richard A. Gibbs, Hiram Scott College, Nebraska.*

A necessary and sufficient condition that a solution exist is that n be even.

Sufficiency is easy. If $n=2k$, consider the cardboard as consisting of k^2 2×2 squares. Simply place a match on each of the four segments adjacent to the center point of each 2×2 square.

For necessity, assume a solution exists for an $n\times n$ sheet of cardboard. To each unit square correspond the point at its center. Connect two points if their corresponding squares share a match. By the hypotheses, every point will be joined to exactly two others. Therefore, according to a basic result of Graph Theory, the resulting graph will be a collection of disjoint cycles. Each cycle will enclose a polygonal region whose sides are either horizontal or vertical line segments. Consequently, since the length of each segment is an integer, the area of each polygonal region will be an integer. By Pick's theorem (a beautiful result familiar to anyone who has played with a geo-board) the area of the ith polygonal region is

$$A = \tfrac{1}{2}P_i + I_i - 1$$

where there are P_i points on the perimeter and I_i points in the interior of the ith polygonal region. Since each area is an integer, each P_i is even. As each point is on exactly one perimeter, the sum of the P_i is the total number of points, n^2. Hence n is even.

II. *Solution by Richard L. Breisch, Pennsylvania State University.*

A generalization of the stated problem will be demonstrated. Let the cardboard be an $m\times n$ rectangle. The problem of covering the cardboard in the stated manner has a solution if and only if m and $n\geqq 2$, and m and n are not both odd.

An alternative representation of the problem will be used to demonstrate this. Consider the $m\times n$ array of the center points of the little squares. If two edge-adjacent squares have a match on their mutual edge, connect the centers of these squares with a line segment. Since each little square has exactly two of its sides covered by matches, in the alternative representation, there are exactly two line segments from each point in the array. Hence each connected set of line segments forms a polygon, and the $m\times n$ array is covered by a collection of polygons. Each polygon must have an even number of horizontal segments and an even number of vertical segments. Since there are $m\cdot n$ segments, m and n cannot both be odd integers.

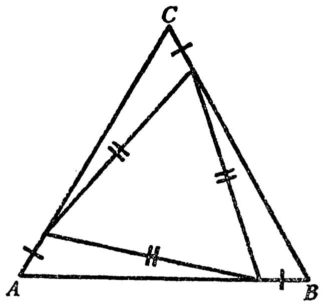
Suppose m is even. Then the $m\times n$ array can be covered with $m/2$ rectangular polygons each of which has dimensions 1 segment by n segments. The arrangement of matches in the original representation is easily derived from this representation.

Also solved by Dan Bean, Dave Harris and E. F. Schmeichel (jointly), College of Wooster, Ohio; Thomas A. Brown, Santa Monica, California; Melvin H. Davis, New York University; Roger Engle and Necdet Ucoluk (jointly), Clarion State College Pennsylvania; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Heiko Harborth, Braunschweig, Germany; Herbert R. Leifer, Pittsburgh, Pennsylvania; Joseph V. Michalowicz, Catholic University of America; George A. Novacky, Jr., University of Pittsburgh; J. W. Pfaendtner, University of Michigan; Sally Ringland, Shippenville, Pennsylvania; Rina Rubenfeld, New York City Community College; E. P. Starke, Plainfield, New Jersey; and the proposer.

Comment on Problem 574

574. [March and November, 1970] *Proposed by NSF Class at University of California at Berkeley.*

Show that the triangle ABC is equilateral.



Comment by K. R. S. Sastry, Makele, Ethiopia.

Michael Goldberg's solution to the case of "Nested Equilateral Triangles" is so general that we can easily extend it to the case of "Nested Regular Polygons of n Sides." By starting with $A > (n-2)\,180/n$ we arrive at the same contradiction $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1$.

Comment on T46

T46. [November, 1961] The most common proof of the theorem, "The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides", involves drawing a parallel line and using similar triangles. Devise a shorter proof using areas.

[*Submitted by Robert P. Goldberg*]

Comment by W. C. McDaniel, Southern Illinois University, Carbondale.

In the solution given, use is made of the fact that the perpendicular from C to the opposite side is the altitude of both smaller triangles so that their areas are proportional to the bases, m and n. I note that since D is on the angle bisector the perpendicular segments to the sides of the angle BCA are equal and that these are also the altitudes of the two triangles. Thus the areas are also proportional to the bases, a and b. Consequently, $m/n = a/b$.

Comment on Q443

Q443. [November, 1968] Take the digits of a number expressed in any given base and permute them in any order. Prove that the difference between the two numbers is divisible by a number one less than the base.

[*Submitted by Michael Garrick and Mrs. Jack Lochhead*]

Comment by Charles W. Trigg, San Diego, California.

It is well known and easily proven that in any base b, a number is congruent to the sum of its digits modulo $b-1$. Since the number N and the permutation P have the same digit sum S, then $N \equiv S$ and $P \equiv S \mod(b-1)$, so $N-P \equiv 0 \mod(b-1)$.

Comment on Q505

Q505. [January, 1971] Solve the differential equation

$$(x-a)(x-b)y'' + 2(2x-a-b)y' + 2y = 0.$$

[*Submitted by Gregory Wulczyn*]

Comment by M. S. Klamkin, Ford Motor Company.

The problem can be easily extended and solved to the differential equation

$$(x-a)(x-b)y'' + n(2x-a-b)y' + n(n-1)y = 0 \qquad (n = 2, 3, 4, \cdots).$$

Letting $y = D^{n-2}z$, the differential equation can be rewritten as

$$D^n\{z(x-a)(x-b)\} = 0.$$

Whence,

$$z = \frac{1}{(x-a)(x-b)}\{A_0 + A_1x + \cdots + A_{n-1}x^{n-1}\}$$

$$= B_0 + B_1x + \cdots + B_{n-3}x^{n-3} + \frac{A}{x-a} + \frac{B}{x-b}\cdot$$

Finally,

$$y = D^{n-2}\left\{\frac{A}{x-a} + \frac{B}{x-b}\right\} = \frac{A'}{(x-a)^{n-1}} + \frac{B'}{(x-b)^{n-1}}\cdot$$

A similar comment was submitted by Leonard J. Putnick, Siena College, New York.

Comment on Q507

Q507. [January, 1971] For what values of N is $7N+55$ a factor of N^2-71?

[*Submitted by David L. Silverman*]

Comment by Charles W. Trigg, San Diego, California.

An alternate approach involving little computation is:

$$(n^2-71)/(7n+55) = (1/49)(49n^2-3479)/(7n+55)$$
$$= (1/49)[7n-55-2(227)/(7n+55)].$$

Hence $7n+55 = \pm(1, 2, 227, \text{ or } 454)$. Only -1 and $+454$ lead to integral values, namely: $n=-8$ and 57.

Comment on Q512

Q512. [January, 1971] Find the solution of

$$x^{(x+1)} + x^x - 1 = 0.$$

[*Submitted by Patricia LaFratta*]

Comment by Benjamin L. Schwartz, McLean, Virginia.

The answer may be correct, but the proof definitely is not. The function $f(x) = x\ln x + \ln(x+1)$ is not monotone for $x > 0$. Consider $f'(x) = 1 + \ln x + 1/(1+x)$. This is obvious negative for small enough, positive x. Interpolation gives $x_0 = 0.154742$ as the only zero of f'. Hence for $x < x_0$, f is decreasing. The published solution is the only one in the open interval $x > x_0$. However, there may be another for $x < x_0$. Letting x approach zero, we find $\lim_{x\to 0} f(x) = 0$ (using l'Hospital's rule, if necessary, to evaluate the indeterminate form $0 \ln 0$). Hence if x is permitted to assume the value 0, with f being interpreted as taking its limiting value, we do indeed have a second solution, viz., $x = 0$.

Comment on Q513

Q513. [March, 1971] Twelve numbers are in arithmetical progression such that $a_k + d = a_{k+1}$. Find the volume of the tetrahedron with vertices (a_1^2, a_2^2, a_3^2), (a_4^2, a_5^2, a_6^2), (a_7^2, a_8^2, a_9^2), $(a_{10}^2, a_{11}^2, a_{12}^2)$

[*Submitted by Charles W. Trigg*]

Comment by Sid Spital, California State College at Hayward.

The answer also follows from the determinant (or scalar triple product) expression for tetrahedron volume:

$$\pm V = \frac{1}{6}\begin{vmatrix} a_4^2 - a_1^2 & a_5^2 - a_2^2 & a_6^2 - a_3^2 \\ a_7^2 - a_1^2 & a_8^2 - a_2^2 & a_9^2 - a_3^2 \\ a_{10}^2 - a_1^2 & a_{11}^2 - a_2^2 & a_{12}^2 - a_3^2 \end{vmatrix} = \frac{(3d)^3}{6}\begin{vmatrix} a_4 + a_1 & a_5 + a_2 & a_6 + a_3 \\ a_7 + a_1 & a_8 + a_2 & a_9 + a_3 \\ a_{10} + a_1 & a_{11} + a_2 & a_{12} + a_3 \end{vmatrix}.$$

Subtracting the second row from the third and then the first row from the second shows that $V = 0$.

ANSWERS

A528. $781 \equiv 7 \pmod 9$ and $965 \equiv 2 \pmod 9$. Thus 753, $A65 \equiv 5 \pmod 9$ and $A = 6$.

A529. Let $x = 1 - t$. The given integral becomes

$$\int_0^1 (1 - x^n)/(1 + x)dx = \int_0^1 1 + x + \cdots + x^{n-1}dx.$$

A530. Since

$$\binom{a-2}{5} = \frac{(a-1)(x+1)(a^3 - 4a)}{120}$$

is an integer and since both primes are not divisible by 2, 3, 4 or 5 we see that 120 must divide a^3-4a.

A531. The answer is no, since such a set could be interpreted as the graph of a continuous contraction of a closed disc on its boundary.

(Quickies on pages 287–288.)

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