

BOUNDARY FUNCTIONS AND SETS OF CURVILINEAR CONVERGENCE FOR CONTINUOUS FUNCTIONS

BY

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Let D be the open unit disk in the complex plane, and let C be its boundary, the unit circle. If $x \in C$, then by an *arc at x* we mean a simple arc γ with one end point at x such that $\gamma - \{x\} \subseteq D$. If f is a function mapping D into some metric space M , then the *set of curvilinear convergence* of f is defined to be

$\{x \in C: \text{there exists an arc } \gamma \text{ at } x \text{ and there exists a point } p \in M \text{ such that } f(z) \rightarrow p \text{ as } z \rightarrow x \text{ along } \gamma\}$.

If ϕ is a function whose domain is a subset E of the set of curvilinear convergence of f , then ϕ is called a *boundary function* for f if, and only if, for each $x \in E$ there exists an arc γ at x such that $f(z) \rightarrow \phi(x)$ as $z \rightarrow x$ along γ . Let S be another metric space. We shall say that a function ϕ is of *Baire class $\leq 1(S, M)$* if

- (i) domain $\phi = S$,
- (ii) range $\phi \subseteq M$, and
- (iii) there exists a sequence $\{\phi_n\}$ of continuous functions, each mapping S into M , such that $\phi_n \rightarrow \phi$ pointwise on S .

We shall say that ϕ is of *honorary Baire class $\leq 2(S, M)$* if

- (i) domain $\phi = S$,
- (ii) range $\phi \subseteq M$, and
- (iii) there exists a countable set $N \subseteq S$ and there exists a function ψ of Baire class $\leq 1(S, M)$ such that $\phi(x) = \psi(x)$ for every $x \in S - N$.

It is known that if f is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$, and any boundary function for f is of honorary Baire class $\leq 2(C, \text{Riemann sphere})$. (See [3], [4], [5], [6], [9].) J. E. McMillan [6] posed the following problem. If A is a given set in C of type $F_{\sigma\delta}$, and if ϕ is a function of honorary Baire class $\leq 2(A, \text{Riemann sphere})$, does there always exist a continuous function f mapping D into the Riemann sphere such that A is the set of curvilinear convergence of f and ϕ is a boundary function for f ? The purpose of this paper is to give an affirmative answer to McMillan's question. However, the corresponding question for real-valued functions remains open. (See Problems 1 and 2 at the end of this paper.) In proving our result, we first give a proof under the assumption that ϕ is a bounded complex-valued function, and we then use a certain device to transfer the theorem to the Riemann sphere. As we shall indicate in an appendix, the same device can be

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used to transfer certain results concerning real-valued functions of the first Baire class to the case of functions taking values on the Riemann sphere.

Our proof is divided into several major steps, which are labeled (A), (B), (C), etc. The proofs of some of the major steps are divided into smaller steps, which are labeled (I), (II), (III), etc. The results (A) and (B) are taken from the author's doctoral dissertation [5].

Throughout this paper we shall use the following notation. R denotes the set of real numbers, S^2 denotes the Riemann sphere, and R^n denotes n -dimensional Euclidean space. Points in R^n will be written in the form $\langle x_1, x_2, \dots, x_n \rangle$ (rather than (x_1, x_2, \dots, x_n)) in order to avoid confusion with open intervals of real numbers in the case $n=2$. The empty set will be denoted by \emptyset . When we speak of a complex-valued function, we mean a function taking only *finite* complex values. The closure of a set E will be denoted either by \bar{E} or by $\text{Cl } E$. If I is an interval of real numbers, then I^* denotes the interior of I . If p is a point of some metric space and $r \in (0, +\infty)$, then $S(r, p)$ denotes the set of all points of the space having distance (strictly) less than r from p .

We define

$$\begin{aligned} Q &= \{\langle x, y \rangle \in R^2 : -1 \leq x \leq 1, 0 < y \leq 1\}, \\ X &= \{\langle x, 0 \rangle : -1 < x < 1\}, \\ H &= \{\langle x, y \rangle \in R^2 : y > 0\}. \end{aligned}$$

It will be convenient to identify $\langle x, 0 \rangle$ with the real number x , and X with $(-1, 1)$. If f is a complex-valued function defined in Q , then we shall understand the set of curvilinear convergence of f to mean the set of all $x \in X$ for which there exists an arc γ at x (contained in the interior of Q except for its end point at x) such that f approaches a finite limit along γ . If $a \in X$, $\varepsilon > 0$, and $0 < \theta < \frac{1}{2}\pi$, then we let

$$s(a, \varepsilon, \theta) = \{\langle x, y \rangle \in R^2 : 0 < y < \varepsilon, a - y \text{ ctn } \theta < x < a + y \text{ ctn } \theta\}.$$

Thus $s(a, \varepsilon, \theta)$ is the interior of an isosceles triangle in H with apex at a .

(A) If $A \subseteq X$ is a set of type $F_{\sigma\delta}$, then there exists a bounded continuous real-valued function g defined in Q such that

- (i) for each $x \in A$, $g(z) \rightarrow 0$ as z approaches x through $s(x, 1, \frac{3}{8}\pi)$, and
- (ii) if $x \in X$, and if there exists an arc γ at x such that $g(z) \rightarrow 0$ as z approaches x along γ , then $x \in A$.

(I) Let E_1 and E_2 be two sets on the real line. A point $p \in R$ will be called a *splitting point* for E_1 and E_2 if either

- (i) $x_1 \leq p$ for all $x_1 \in E_1$ and $p \leq x_2$ for all $x_2 \in E_2$, or
- (ii) $x_2 \leq p$ for all $x_2 \in E_2$ and $p \leq x_1$ for all $x_1 \in E_1$.

We will say that E_1 and E_2 *split* if and only if there exists a splitting point for E_1 and E_2 .

(II) By a *special family* we mean a family \mathcal{F} of subsets of X such that

- (i) \mathcal{F} is nonempty,

- (ii) \mathcal{F} is countable,
- (iii) each member of \mathcal{F} is compact,
- (iv) if $E, F \in \mathcal{F}$, then either $E=F$, $E \cap F = \emptyset$, or E and F split.

(III) If $E \subseteq X$ is an F_σ set, then there exists a special family \mathcal{F} such that $E = \bigcup \mathcal{F}$.

Proof. We can write $E = \bigcup_{n=1}^\infty A_n$, where $A_1 = \emptyset$, A_n is closed, and $A_n \subseteq A_{n+1}$ for all n . Observe that if I is any open interval contained in X , then there exists a countable family $\{J_n\}_{n=1}^\infty$ of compact intervals contained in X such that $I = \bigcup_{n=1}^\infty J_n$, and $n \neq m$ implies that J_n and J_m split. Since $X - A_n$ is a countable disjoint union of open intervals, it follows that we can choose (for each n) a family

$$\{I_{n,j}\}_{j=1}^\infty$$

of compact intervals such that $X - A_n = \bigcup_{j=1}^\infty I_{n,j}$, and $j \neq k$ implies that $I_{n,j}$ and $I_{n,k}$ split. Let

$$\mathcal{F} = \{A_1\} \cup \{I_{n,j} \cap A_{n+1} : n = 1, 2, \dots; j = 1, 2, \dots\}.$$

Then \mathcal{F} is a countable family of compact sets, and

$$\begin{aligned} E &= \bigcup_{n=1}^\infty A_n = A_1 \cup \bigcup_{n=1}^\infty A_{n+1} \cap (X - A_n) \\ &= A_1 \cup \bigcup_{n=1}^\infty \bigcup_{j=1}^\infty A_{n+1} \cap I_{n,j} \\ &= \bigcup \mathcal{F}. \end{aligned}$$

Let F_1 and F_2 be any two distinct members of \mathcal{F} . If either F_1 or F_2 is $A_1 = \emptyset$, then F_1 and F_2 are automatically disjoint. If neither F_1 nor F_2 is A_1 , then we can write

$$\begin{aligned} F_1 &= I_{n(1),j(1)} \cap A_{n(1)+1}, \\ F_2 &= I_{n(2),j(2)} \cap A_{n(2)+1}. \end{aligned}$$

If $n(1) < n(2)$, then $n(1) + 1 \leq n(2)$, so

$$F_2 = I_{n(2),j(2)} \cap A_{n(2)+1} \subseteq X - A_{n(2)} \subseteq X - A_{n(1)+1} \subseteq X - F_1,$$

and therefore F_1 and F_2 are disjoint. If $n(2) < n(1)$, a similar argument shows that F_1 and F_2 are disjoint. Now suppose $n(1) = n(2)$. Then, since $F_1 \neq F_2$, we have $j(1) \neq j(2)$. So $I_{n(1),j(2)} = I_{n(2),j(2)}$ and $I_{n(1),j(1)}$ split, and consequently F_1 and F_2 split. We have shown that any two distinct members of \mathcal{F} either split or are disjoint, so \mathcal{F} is a special family.

(IV) Let $A \subseteq X$ be a set of type $F_{\sigma\delta}$. Then there exists a sequence of special families $\{\mathcal{F}_n\}_{n=1}^\infty$ such that

- (i) $A = \bigcap_{n=1}^\infty (\bigcup \mathcal{F}_n)$,
- (ii) if $n \geq 1$ and $E \in \mathcal{F}_{n+1}$, then there exists $F \in \mathcal{F}_n$ with $E \subseteq F$.

Proof. There exist F_σ sets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ such that $A = \bigcap_{n=1}^\infty A_n$. By (III), we can choose (for each n) a special family \mathcal{E}_n such that $A_n = \bigcup \mathcal{E}_n$. Let $\mathcal{F}_1 = \mathcal{E}_1$. For $n \geq 1$, let

$$\mathcal{F}_{n+1} = \{F \cap E : F \in \mathcal{F}_n \text{ and } E \in \mathcal{E}_{n+1}\}.$$

By induction on n , one can show that each \mathcal{F}_n is a special family and that $A_n = \bigcup \mathcal{F}_n$. It is clear that the other conditions are satisfied.

(V) Suppose that J is a nonempty interval with $\bar{J} \subseteq X$, and let a, b ($a \leq b$) be the end points of J . By Trap (J, ε, θ) (where $\theta \in (0, \frac{1}{2}\pi$) and $\varepsilon > 0$) we mean the trapezoid-shaped open set defined by

$$\text{Trap}(J, \varepsilon, \theta) = \{ \langle x, y \rangle : 0 < y < \varepsilon, a + y \text{ctn } \theta < x < b - y \text{ctn } \theta \}.$$

For $\theta \in (0, \frac{1}{2}\pi)$ let Tri (J, θ) be the closed triangular area defined by

$$\text{Tri}(J, \theta) = \{ \langle x, y \rangle : y \geq 0, a + y \text{ctn } \theta \leq x \leq b - y \text{ctn } \theta \}.$$

If K is a nonempty compact subset of X , let $J(K)$ be the smallest closed interval containing K . If $\varepsilon > 0$ and $0 < \beta < \alpha < \frac{1}{2}\pi$, then we define

$$B(K, \varepsilon, \alpha, \beta) = \text{Trap}(J(K), \varepsilon, \alpha) - \bigcup_{I \in \mathcal{I}} \text{Tri}(I, \beta),$$

where \mathcal{I} denotes the (possibly empty) set of disjoint nonempty open intervals whose union is $J(K) - K$.

We state without proof the following readily verifiable facts ((VI) through (XVIII)).

(VI) $s(x, \varepsilon, \theta)$ is an open subset of H .

(VII) $\text{Cl}[s(x, \varepsilon, \theta)] \cap X = \{x\}$.

(VIII) If $\varepsilon < \varepsilon'$ and $\theta' < \theta$, then $\text{Cl}[s(x, \varepsilon, \theta)] \cap H \subseteq s(x, \varepsilon', \theta')$.

(IX) If $x \neq y$ and ε, θ are given, then there exists $\delta > 0$ such that, for every $\eta \leq \delta$, $s(x, \varepsilon, \theta)$ and $s(y, \eta, \theta)$ are disjoint.

(X) $B(K, \varepsilon, \alpha, \beta)$ is an open subset of H .

(XI) If K_1 and K_2 split, then, for any $\varepsilon_1, \varepsilon_2, \alpha$, and β , $B(K_1, \varepsilon_1, \alpha, \beta)$ and $B(K_2, \varepsilon_2, \alpha, \beta)$ are disjoint.

(XII) If K_1 and K_2 are disjoint compact subsets of X , and if $\varepsilon, \alpha, \beta$ are given, then there exists $\delta > 0$ such that for every $\eta \leq \delta$, $B(K_1, \varepsilon, \alpha, \beta)$ and $B(K_2, \eta, \alpha, \beta)$ are disjoint.

(XIII) $\text{Cl}[B(K, \varepsilon, \alpha, \beta)] \cap X \subseteq K$.

(XIV) Suppose that $K_1 \subseteq K$, $\varepsilon > \varepsilon_1 > 0$, and $0 < \beta < \beta_1 < \alpha_1 < \alpha < \pi/2$. Then $\text{Cl}[B(K_1, \varepsilon_1, \alpha_1, \beta_1)] \cap H \subseteq B(K, \varepsilon, \alpha, \beta)$.

(XV) Suppose that $\alpha < \theta < \frac{1}{2}\pi$ and $x \notin J(K)^*$. Then, for any $\varepsilon, \varepsilon_1$, and β , $B(K, \varepsilon, \alpha, \beta)$ and $s(x, \varepsilon_1, \theta)$ are disjoint.

(XVI) Suppose that $x \notin K$ and that $\varepsilon, \alpha, \beta, \theta$ are given. Then there exists $\delta > 0$ such that for every $\eta \leq \delta$, $s(x, \eta, \theta)$ and $B(K, \varepsilon, \alpha, \beta)$ are disjoint.

(XVII) Suppose that $x \notin K$ and that $\delta, \alpha, \beta, \theta$ are given. Then there exists $\varepsilon > 0$ such that for every $\eta \leq \varepsilon$, $s(x, \delta, \theta)$ and $B(K, \eta, \alpha, \beta)$ are disjoint.

(XVIII) Suppose that $x \in K \cap J(K)^*$ and $0 < \beta < \alpha < \theta < \frac{1}{2}\pi$. Let ε be given. Then there exists $\delta > 0$ such that for every $\eta \leq \delta$, $\text{Cl}[s(x, \eta, \theta)] \cap H \subseteq B(K, \varepsilon, \alpha, \beta)$.

(XIX) If \mathcal{F} is a special family, let \mathcal{F}^2 be the set of all members of \mathcal{F} that have two or more points, and let $E(\mathcal{F})$ be the set of all end points of intervals $J(F)$, where $F \in \mathcal{F}$ and $F \neq \emptyset$.

Suppose that $0 < \beta < \alpha < \theta < \frac{1}{2}\pi$, and that \mathcal{F} is a special family. By a pair of special α, β, θ functions for \mathcal{F} , I mean a pair (ε, δ) , where ε and δ are positive real-valued functions, the domain of ε is $E(\mathcal{F})$, the domain of δ is \mathcal{F}^2 , and

- (i) for each $\eta > 0$, there exist at most finitely many $F \in \mathcal{F}^2$ such that $\delta(F) \geq \eta$;
- (ii) for each $\eta > 0$, there exist at most finitely many $x \in E(\mathcal{F})$ such that $\varepsilon(x) \geq \eta$;
- (iii) if $x, x' \in E(\mathcal{F})$ and $x \neq x'$, then $s(x, \varepsilon(x), \theta)$ and $s(x', \varepsilon(x'), \theta)$ are disjoint;
- (iv) if $F, K \in \mathcal{F}^2$ and $F \neq K$, then $B(F, \delta(F), \alpha, \beta)$ and $B(K, \delta(K), \alpha, \beta)$ are disjoint;

(v) if $x \in E(\mathcal{F})$ and $F \in \mathcal{F}^2$, then $s(x, \varepsilon(x), \theta)$ and $B(F, \delta(F), \alpha, \beta)$ are disjoint.

(XX) Let \mathcal{F} be a special family and suppose that $0 < \beta < \alpha < \theta < \frac{1}{2}\pi$. Then there exists a pair of special α, β, θ functions for \mathcal{F} .

Though a formal proof of this statement is lengthy, it requires no originality, so we omit the details. The idea is to arrange the members of \mathcal{F} in a finite or infinite sequence F_1, F_2, F_3, \dots , and then define ε and δ inductively. One makes use of statements (IX), (XI), (XII), (XV), (XVI), (XVII).

(XXI) Let \mathcal{F} be a special family, and suppose $0 < \beta < \alpha < \theta < \frac{1}{2}\pi$. Let (ε, δ) be a pair of special α, β, θ functions for \mathcal{F} . If ε_1, δ_1 are two real-valued functions having domains $E(\mathcal{F})$ and \mathcal{F}^2 respectively, and if

$$0 < \varepsilon_1(x) \leq \varepsilon(x) \quad \text{for all } x \in E(\mathcal{F}),$$

$$0 < \delta_1(F) \leq \delta(F) \quad \text{for all } F \in \mathcal{F}^2,$$

then $(\varepsilon_1, \delta_1)$ is a pair of special α, β, θ functions for \mathcal{F} .

The proof of this statement is trivial.

(XXII) We now proceed to the proof of statement (A) itself. Let A be our given $F_{\sigma\delta}$ set. By (IV), we can choose a sequence of special families $\{\mathcal{F}_n\}_{n=1}^\infty$ such that $A = \bigcap_{n=1}^\infty (\bigcup \mathcal{F}_n)$, and for each $K \in \mathcal{F}_{n+1}$ there exists $F \in \mathcal{F}_n$ with $K \subseteq F$.

Let $\{\beta_n\}_{n=1}^\infty$ be a strictly increasing sequence in $(0, \frac{1}{8}\pi)$ converging to $\frac{1}{8}\pi$.

Let $\{\alpha_n\}_{n=1}^\infty$ be a strictly decreasing sequence in $(\frac{1}{8}\pi, \frac{1}{4}\pi)$ converging to $\frac{1}{8}\pi$.

Let $\{\theta_n\}_{n=1}^\infty$ be a strictly increasing sequence in $(\frac{1}{4}\pi, \frac{3}{8}\pi)$ converging to $\frac{3}{8}\pi$.

Let $E_n = E(\mathcal{F}_n)$.

Let $(\varepsilon(1, \cdot), \delta(1, \cdot))$ be any pair of special $\alpha_1, \beta_1, \theta_1$ functions for \mathcal{F}_1 .

Now suppose that for each $k \leq n$ we have chosen a pair of special $\alpha_k, \beta_k, \theta_k$ functions $(\varepsilon(k, \cdot), \delta(k, \cdot))$ for \mathcal{F}_k in such a way that

- (i) whenever $1 \leq k \leq n-1$, $x \in E_{k+1}$, $F \in \mathcal{F}_k$, and $x \in F \cap J(F)^*$, then

$$\text{Cl } [s(x, \varepsilon(k+1, x), \theta_{k+1})] \cap H \subseteq B(F, \delta(k, F), \alpha_k, \beta_k);$$

- (ii) whenever $1 \leq k \leq n-1$, $x \in E_{k+1}$, and $x \in E_k$, then

$$\text{Cl } [s(x, \varepsilon(k+1, x), \theta_{k+1})] \cap H \subseteq s(x, \varepsilon(k, x), \theta_k);$$

- (iii) whenever $1 \leq k \leq n-1$, $K \in (\mathcal{F}_{k+1})^2$, $F \in (\mathcal{F}_k)^2$, and $K \subseteq F$, then

$$\text{Cl } [B(K, \delta(k+1, K), \alpha_{k+1}, \beta_{k+1})] \cap H \subseteq B(F, \delta(k, F), \alpha_k, \beta_k).$$

Then we construct $(\varepsilon(n+1, \cdot), \delta(n+1, \cdot))$ as follows. Let (ε, δ) be any pair of special $\alpha_{n+1}, \beta_{n+1}, \theta_{n+1}$ functions for \mathcal{F}_{n+1} . If $x \in E_{n+1} - E_n$, then for some unique $F \in (\mathcal{F}_n)^2, x \in F \cap J(F)^*$. By (XVIII), we can choose $\xi(x) > 0$ so that $\eta \leq \xi(x)$ implies

$$\text{Cl} [s(x, \eta, \theta_{n+1})] \cap H \subseteq B(F, \delta(n, F), \alpha_n, \beta_n).$$

We set $\varepsilon(n+1, x) = \min \{ \varepsilon(x), \xi(x) \}$. On the other hand, if $x \in E_{n+1} \cap E_n$, then we set $\varepsilon(n+1, x) = \min \{ \varepsilon(x), \frac{1}{2}\varepsilon(n, x) \}$.

If $K \in (\mathcal{F}_{n+1})^2$, then there exists a unique $F \in (\mathcal{F}_n)^2$ with $K \subseteq F$. Set

$$\delta(n+1, K) = \min \{ \delta(K), \frac{1}{2}\delta(n, F) \}.$$

By (XXI), $(\varepsilon(n+1, \cdot), \delta(n+1, \cdot))$ is a pair of special $\alpha_{n+1}, \beta_{n+1}, \theta_{n+1}$ functions for \mathcal{F}_{n+1} , and, by (VIII) and (XIV), conditions (i), (ii), (iii) are still satisfied when n is replaced by $n+1$. Thus we can inductively construct a pair $(\varepsilon(n, \cdot), \delta(n, \cdot))$ of special $\alpha_n, \beta_n, \theta_n$ functions for \mathcal{F}_n in such a way that (i), (ii), and (iii) are satisfied for every n .

Let

$$U_n = \left[\bigcup_{x \in E_n} s(x, \varepsilon(n, x), \theta_n) \right] \cup \left[\bigcup_{F \in (\mathcal{F}_n)^2} B(F, \delta(n, F), \alpha_n, \beta_n) \right].$$

Then U_n is open. For fixed n , all the various sets $s(x, \varepsilon(n, x), \theta_n)$ ($x \in E_n$) and $B(F, \delta(n, F), \alpha_n, \beta_n)$ ($F \in (\mathcal{F}_n)^2$) are open and pairwise disjoint, so that every component of U_n is contained in one of the sets $s(x, \varepsilon(n, x), \theta_n)$ ($x \in E_n$) or $B(F, \delta(n, F), \alpha_n, \beta_n)$ ($F \in (\mathcal{F}_n)^2$). It therefore follows from (VII) and (XIII) that if W is any component of U_n , then

$$(1) \quad \overline{W} \cap X \subseteq \bigcup \mathcal{F}_n.$$

From conditions (i) and (ii) in the definition of a pair of special α, β, θ functions, it follows that

$$\overline{U}_n \cap H = \left[\bigcup_{x \in E_n} \text{Cl} [s(x, \varepsilon(n, x), \theta_n)] \cap H \right] \cup \left[\bigcup_{F \in (\mathcal{F}_n)^2} \text{Cl} [B(F, \delta(n, F), \alpha_n, \beta_n)] \cap H \right].$$

Consequently, conditions (i), (ii), (iii) in our inductive construction of $(\varepsilon(n, \cdot), \delta(n, \cdot))$ (together with the fact that $x \in E_{n+1} - E_n$ implies $x \in F \cap J(F)^*$ for some $F \in (\mathcal{F}_n)^2$) imply that $\overline{U}_{n+1} \cap H \subseteq U_n$ for every n .

By Urysohn's Lemma there exists a continuous function $g_n: H \rightarrow [0, 1]$ such that $g_n(z) = 1$ for $z \in H - U_n$ and $g_n(z) = 0$ for $z \in \overline{U}_{n+1} \cap H$. Let

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n g_n(z).$$

Then $0 \leq g(z) \leq 1$, and the series converges uniformly, so g is continuous in H .

If $z \in H - U_n$, then $z \in H - U_m$ for every $m \geq n$, so that $1 = g_n(z) = g_{n+1}(z) = g_{n+2}(z) = \dots$, and hence

$$(2) \quad g(z) \geq \sum_{m=n}^{\infty} \left(\frac{1}{2}\right)^m = \left(\frac{1}{2}\right)^{n-1} \quad (z \in H - U_n).$$

Also, if $z \in U_{n+1}$, then $z \in U_1, U_2, \dots, U_{n+1}$, so that $0 = g_1(z) = g_2(z) = \dots = g_n(z)$, and

$$(3) \quad g(z) \leq \sum_{m=n+1}^{\infty} \left(\frac{1}{2}\right)^m = \left(\frac{1}{2}\right)^n \quad (z \in U_{n+1}).$$

Let $x_0 \in A$ be given. We must show that $g(z) \rightarrow 0$ as z approaches x_0 through $s(x_0, 1, \frac{3}{8}\pi)$. Take any natural number n . Since $x_0 \in \bigcup \mathcal{F}_{n+1}$, it follows that either $x_0 \in E_{n+1}$ or else $x_0 \in F \cap J(F)^*$ for some $F \in (\mathcal{F}_{n+1})^2$. In the first case, set $\eta = \varepsilon(n+1, x_0)$. In the second case, (XVIII) shows that we can choose $\eta > 0$ small enough so that

$$s(x_0, \eta, \frac{3}{8}\pi) \subseteq B(F, \delta(n+1, F), \alpha_{n+1}, \beta_{n+1}).$$

Suppose $\langle x, y \rangle \in s(x_0, 1, \frac{3}{8}\pi)$ and $y < \eta$. Then, in the first case,

$$\langle x, y \rangle \in s(x_0, \eta, \frac{3}{8}\pi) \subseteq s(x_0, \varepsilon(n+1, x_0), \theta_{n+1}) \subseteq U_{n+1},$$

and, in the second case,

$$\langle x, y \rangle \in s(x_0, \eta, \frac{3}{8}\pi) \subseteq B(F, \delta(n+1, F), \alpha_{n+1}, \beta_{n+1}) \subseteq U_{n+1}.$$

Thus, referring to (3), we see that $g(x, y) \leq (\frac{1}{2})^n$ whenever $\langle x, y \rangle \in s(x_0, 1, \frac{3}{8}\pi)$ and $y < \eta$. Therefore $g(z) \rightarrow 0$ as z approaches x_0 through $s(x_0, 1, \frac{3}{8}\pi)$.

Let x_1 be a point of X , and assume there exists an arc γ at x_1 such that $g(z) \rightarrow 0$ as z approaches x_1 along γ . Then γ has a subarc γ' with one end point at x_1 such that $\gamma' - \{x_1\} \subseteq g^{-1}((-\frac{1}{2})^n, (\frac{1}{2})^n)$. By (2), $\gamma' - \{x_1\} \subseteq U_n$. Therefore, by (1), $x_1 \in \bigcup \mathcal{F}_n$. Since n is arbitrary,

$$x_1 \in \bigcap_{n=1}^{\infty} (\bigcup \mathcal{F}_n) = A.$$

Thus, by restricting g to Q we obtain the desired function.

(B) Let A be a subset of X of type $F_{\sigma\delta}$, and let ϕ be a bounded complex-valued function of honorary Baire class $\leq 2(A, R^2)$. Then there exists a bounded continuous complex-valued function h defined in Q such that, for each $x \in A$, there exists an arc γ at x with $\gamma - \{x\} \subseteq s(x, 1, \frac{3}{8}\pi)$ and

$$\lim_{z \rightarrow x; z \in \gamma} h(z) = \phi(x).$$

(I) Let I be a bounded open interval in R , and let $f: I \rightarrow R$ be a bounded, strictly increasing function. Then there exists a continuous, weakly increasing function $f^*: R \rightarrow R$ such that $f^*(f(x)) = x$ for every $x \in I$. (This result is probably not new, but I do not know of a reference for it, so I am obliged to prove it here.)

Proof. Let $Z = f(I)$, let $c = \inf Z$, and let $d = \sup Z$. Observe that $Z \subseteq (c, d)$, and that $f^{-1}: Z \rightarrow I$ is strictly increasing. I assert that for each $x \in (c, d)$

$$(4) \quad \sup f^{-1}((c, x] \cap Z) = \sup f^{-1}((c, x) \cap Z).$$

If $x \notin Z$, the equation is trivial. Suppose $x \in Z$. Then

$$c < y < f^{-1}(x) \Rightarrow (f(y) < x \text{ and } f(y) \in Z),$$

so that $(c, f^{-1}(x)) \subseteq f^{-1}((c, x) \cap Z)$. Hence

$$\sup f^{-1}((c, x) \cap Z) \geq f^{-1}(x) = \sup f^{-1}((c, x] \cap Z).$$

The opposite inequality is trivial, so (4) is established.

I also assert that for each $x \in (c, d)$

$$(5) \quad \inf f^{-1}((x, d) \cap Z) = \sup f^{-1}((c, x] \cap Z).$$

Obviously,

$$\inf f^{-1}((x, d) \cap Z) \geq \sup f^{-1}((c, x] \cap Z).$$

Take any $y > \sup f^{-1}((c, x] \cap Z)$. If $f(y) \leq x$, then $f(y) \in (c, x] \cap Z$, and so $y \in f^{-1}((c, x] \cap Z)$, a contradiction. Thus $f(y) > x$ and $f(y) \in (x, d) \cap Z$. Therefore $y \in f^{-1}((x, d) \cap Z)$, so that $\inf f^{-1}((x, d) \cap Z) \leq y$. In view of the choice of y , this implies that

$$\inf f^{-1}((x, d) \cap Z) \leq \sup f^{-1}((c, x] \cap Z),$$

and (5) is established.

Define f^* on (c, d) by

$$f^*(x) = \sup f^{-1}((c, x] \cap Z) \quad (x \in (c, d)).$$

It is clear that f^* is weakly increasing and that $f^*(f(x)) = x$ for every $x \in I$. The continuity of f^* can easily be deduced from the equations

$$\sup f^*((c, x)) = f^*(x), \quad \inf f^*((x, d)) = f^*(x),$$

which are established as follows:

$$\begin{aligned} \sup f^*((c, x)) &= \sup_{c < y < x} \sup f^{-1}((c, y] \cap Z) \\ &= \sup f^{-1}((c, x) \cap Z) \\ &= \sup f^{-1}((c, x] \cap Z) \\ &= f^*(x), \\ \inf f^*((x, d)) &= \inf_{x < y < d} \sup f^{-1}((c, y] \cap Z) \\ &= \inf_{x < y < d} \inf f^{-1}((y, d) \cap Z) \\ &= \inf f^{-1}((x, d) \cap Z) \\ &= \sup f^{-1}((c, x] \cap Z) \\ &= f^*(x). \end{aligned}$$

We now extend f^* to all of R by setting

$$\begin{aligned} f^*(x) &= \inf f^*((c, d)) \quad \text{if } x \leq c, \\ f^*(x) &= \sup f^*((c, d)) \quad \text{if } x \geq d, \end{aligned}$$

and we are finished.

(II) Suppose that M is a metric space and that $u: M \rightarrow R$ is a function having the following property. For every sequence $\{p_n\}$ of points of M , every $p \in M$, and every $y \in R \cup \{-\infty, +\infty\}$, if $p_n \rightarrow p$ and $u(p_n) \rightarrow y$ as $n \rightarrow \infty$, then $y \in R$ and $u(p) = y$. Under this hypothesis, u is continuous.

Proof. Let $\{p_n\}$ be any sequence of points in M converging to a point $p \in M$. We have only to show that $u(p_n) \rightarrow u(p)$. But suppose $u(p_n) \not\rightarrow u(p)$. Then there exists a subsequence $\{u(p_{n(k)})\}$ and there exists $y \in R \cup \{-\infty, +\infty\}$ such that $y \neq u(p)$ and $u(p_{n(k)}) \rightarrow y$ as $k \rightarrow \infty$. Since $p_{n(k)} \rightarrow p$ as $k \rightarrow \infty$, this contradicts our hypothesis.

(III) Let $A \subseteq (-1, 1)$ be of type $F_{\sigma\delta}$, and let ψ be a complex-valued function of Baire class $\leq 1(A, R^2)$. Then there exists a sequence $\{g_n\}$ of continuous functions, each mapping R into R^2 , such that for each $x \in A$, $g_n(x) \rightarrow \psi(x)$ as $n \rightarrow \infty$.

Proof. This can be proved in a more general context, as shown in [5]. For a quick proof of the special case stated above, we can refer to a theorem of Bagemihl and McMillan [1, Theorem 2], which tells us that there exist continuous real-valued functions f_1 and f_2 defined in H such that, for each $x \in A$, f_1 has angular limit $\operatorname{Re}(\psi(x))$ at x and f_2 has angular limit $\operatorname{Im}(\psi(x))$ at x . For each $x \in R$, set

$$g_n(x) = f_1\left(x, \frac{1}{n}\right) + if_2\left(x, \frac{1}{n}\right).$$

(IV) Now we proceed to the proof of statement (B). Let ψ be a function of Baire class $\leq 1(A, R^2)$ and let E be a (possibly empty) countable subset of A such that $\phi(x) = \psi(x)$ for each $x \in A - E$. Let N be an infinite countable set with $E \subseteq N \subseteq X$. Let w be a real-valued function defined on N such that $w(s) > 0$ for each $s \in N$ and

$$\sum_{s \in N} w(s) < 2^{1/2} - 1.$$

For each $x \in X = (-1, 1)$, let $N(x) = \{s \in N: -1 < s < x\}$. Define f on $(-1, 1)$ by setting

$$f(x) = x + \sum_{s \in N(x)} w(s).$$

Then f is a bounded, strictly increasing function on $(-1, 1)$, and $|f(x) - x| < 2^{1/2} - 1$. By (I), there exists a continuous, weakly increasing function $f^*: R \rightarrow R$ such that $f^*(f(x)) = x$ for each $x \in (-1, 1)$.

Let

$$H_0 = \{\langle x, y \rangle \in R^2: 0 < y \leq \frac{1}{2}\}.$$

For fixed $\langle x, y \rangle \in H_0$,

$$u - f^*\left(\frac{x - (1 - y)u}{y}\right)$$

is a strictly increasing continuous function of u that approaches $+\infty$ as $u \rightarrow +\infty$

and $-\infty$ as $u \rightarrow -\infty$. Consequently there exists precisely one number $u(x, y)$ that satisfies the equation

$$(6) \quad u(x, y) - f^*\left(\frac{x - (1-y)u(x, y)}{y}\right) = 0.$$

I assert that $u(x, y)$ is a continuous function on H_0 . We show this by using (II). Suppose that $\langle x, y \rangle \in H_0$, $u_0 \in R \cup \{-\infty, +\infty\}$, $\{\langle x_n, y_n \rangle\} \subseteq H_0$, $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$, and $u(x_n, y_n) \rightarrow u_0$. If $u_0 = +\infty$, then, as $n \rightarrow \infty$,

$$\frac{x_n - (1 - y_n)u(x_n, y_n)}{y_n} \rightarrow -\infty,$$

and so

$$u(x_n, y_n) - f^*\left(\frac{x_n - (1 - y_n)u(x_n, y_n)}{y_n}\right) \rightarrow +\infty,$$

which contradicts (6). So $u_0 \neq +\infty$, and a similar argument shows that $u_0 \neq -\infty$. Thus, by (6),

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \left[u(x_n, y_n) - f^*\left(\frac{x_n - (1 - y_n)u(x_n, y_n)}{y_n}\right) \right] \\ &= u_0 - f^*\left(\frac{x - (1 - y)u_0}{y}\right). \end{aligned}$$

Consequently $u_0 = u(x, y)$. By (II), u is continuous.

From (III), there exists a sequence $\{g_n\}$ of continuous complex-valued functions defined on R such that $g_n(x) \rightarrow \psi(x)$ as $n \rightarrow \infty$ for each $x \in A$. For $n \geq 2$, define

$$h_0(x, y) = (yn(n+1) - n)g_n(u(x, y)) + ((n+1) - yn(n+1))g_{n+1}(u(x, y))$$

when $1/(n+1) \leq y \leq 1/n$. Then h_0 is continuous on H_0 . Let $\{s_n\}_{n=1}^\infty$ be all the elements of N , where $n \neq m$ implies $s_n \neq s_m$. Let

$$\begin{aligned} r_n &= \inf_{x > s_n} f(x), \\ l_n &= \sup_{x < s_n} f(x) = f(s_n), \\ z_n &= \phi(s_n) - \psi(s_n) \quad \text{if } s_n \in E, \\ z_n &= 0 \quad \quad \quad \text{if } s_n \notin E. \end{aligned}$$

Notice that $r_n - l_n > 0$. If x and y are real numbers, define $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. For $\langle x, y \rangle \in H_0$, set

$$\Delta_n(x, y) = [(1 - ny) \vee 0] \left[\left(1 - \frac{1}{r_n - l_n} \left| r_n + l_n - 2s_n + 2 \frac{s_n - x}{y} \right| \right) \vee 0 \right] z_n.$$

Then Δ_n is continuous in H_0 . Observe that $\Delta_n(x, y) = 0$ when $y \geq 1/n$. Using this fact, it is easy to show that, if we set

$$h_1 = h_0 + \sum_{n=1}^{\infty} \Delta_n,$$

then h_1 is defined and continuous on H_0 .

Let p be any point of A . The line

$$(7) \quad x = (f(p) - p)y + p$$

passes through $\langle p, 0 \rangle$, and, since $|f(p) - p| < 2^{1/2} - 1 = \text{ctn } \frac{3}{8}\pi$, the part of this line which lies in H_0 is contained in $s(p, 1, \frac{3}{8}\pi)$. We show that h_1 approaches $\psi(p)$ along this line. By substituting $(f(p) - p)y + p$ for x in the expression for $\Delta_n(x, y)$, one obtains

$$(8) \quad \Delta_n(x, y) = [(1 - ny) \vee 0] \cdot \left[\left(1 - \frac{1}{r_n - l_n} \left| r_n + l_n + 2 \left(\frac{1}{y} - 1 \right) (s_n - p) - 2f(p) \right| \right) \vee 0 \right] z_n.$$

If $p \leq s_n$, then $f(p) \leq l_n$, and one can verify directly that (8) vanishes. If $p > s_n$, then $f(p) \geq r_n$, and again one can verify directly that (8) vanishes. Thus $\Delta_n(x, y)$ vanishes along that part of the line (7) which lies in H .

Solving (7) for $f(p)$, we find that, along the given line,

$$f(p) = (x - (1 - y)p) / y,$$

and hence

$$p = f^*(f(p)) = f^*((x - (1 - y)p) / y).$$

Therefore (if $0 < y \leq \frac{1}{2}$) $p = u(x, y)$. Hence, if $\langle x, y \rangle$ satisfies (7), $n \geq 2$, and $1/(n + 1) \leq y \leq 1/n$, then

$$h_0(x, y) = (yn(n + 1) - n)g_n(p) + ((n + 1) - yn(n + 1))g_{n+1}(p),$$

so that $h_0(x, y)$ lies on the line segment joining $g_n(p)$ to $g_{n+1}(p)$. It follows that $h_0(x, y)$ approaches $\psi(p)$ as $\langle x, y \rangle$ approaches p along the line (7). Since each Δ_n vanishes on the part of this line lying in H , $h_1(x, y)$ also approaches $\psi(p)$ along this line.

Let s_m be any point of E . The definition of f shows that

$$|f(x) - x| \leq \sum_{s \in N} w(s)$$

for all x , and from this it easily follows that

$$|r_m - s_m| \leq \sum_{s \in N} w(s), \quad |l_m - s_m| \leq \sum_{s \in N} w(s).$$

Hence

$$\left| \frac{r_m + l_m}{2} - s_m \right| \leq \sum_{s \in N} w(s) < 2^{1/2} - 1 = \text{ctn } \frac{3}{8}\pi.$$

So the part of the line

$$(9) \quad x = \left(\frac{r_m + l_m}{2} - s_m \right) y + s_m$$

that lies in H_0 is contained in $s(s_m, 1, \frac{3}{8}\pi)$. We show that h_1 approaches $\phi(s_m)$ as

$z \rightarrow s_m$ along this line. Substituting the value of x given by (9) into the expression for Δ_n , we obtain

$$(10) \quad \Delta_n(x, y) = [(1-ny) \vee 0] \cdot \left[\left(1 - \frac{1}{r_n - l_n} \left| r_n - r_m + l_n - l_m + 2 \left(\frac{1}{y} - 1 \right) (s_n - s_m) \right| \right) \vee 0 \right] z_n.$$

If $s_m < s_n$, then $l_m < r_m \leq l_n < r_n$, and one can verify that (10) vanishes. If $s_n < s_m$, then $l_n < r_n \leq l_m < r_m$, and again one can verify that (10) vanishes. Thus, for $n \neq m$, $\Delta_n(x, y) = 0$ when $\langle x, y \rangle$ lies on the line (9) and in H .

If we take $n = m$ in (10), we obtain

$$\Delta_m(x, y) = [(1-my) \vee 0] z_m.$$

Therefore $\Delta_m(x, y)$ approaches $z_m = \phi(s_m) - \psi(s_m)$ along the given line.

Take any $\langle x, y \rangle \in H_0$ satisfying (9), and take any a and b satisfying

$$(11) \quad a < s_m < b.$$

Then $f(a) \leq l_m < \frac{1}{2}(r_m + l_m) < r_m \leq f(b)$, so that

$$(f(a) - s_m)y + s_m < x < (f(b) - s_m)y + s_m;$$

from which we deduce that

$$f(a) < (x - (1-y)s_m)/y < f(b).$$

Since f^* is weakly increasing,

$$a = f^*(f(a)) \leq f^*((x - (1-y)s_m)/y) \leq f^*(f(b)) = b.$$

Because a and b were taken to be any two numbers satisfying (11), we conclude that

$$s_m = f^*((x - (1-y)s_m)/y),$$

whence it follows that $u(x, y) = s_m$. Thus

$$h_0(x, y) = (yn(n+1) - n)g_n(s_m) + ((n+1) - yn(n+1))g_{n+1}(s_m)$$

when $1/(n+1) \leq y \leq 1/n$. Consequently $h_0(x, y)$ approaches $\psi(s_m)$ along the line (9); so $h_1(x, y)$ approaches $\psi(s_m) + \phi(s_m) - \psi(s_m) = \phi(s_m)$ along the given line.

We have shown that, for each $x \in A$, there exists a line segment at x , lying in $s(x, 1, \frac{2}{3}\pi)$, such that $h_1(z) \rightarrow \phi(x)$ as $z \rightarrow x$ along the line segment. We do not know that h_1 is bounded, but this is easily patched up. Choose a real number B such that, for all $x \in A$,

$$-B < \operatorname{Re} \phi(x) < B, \quad -B < \operatorname{Im} \phi(x) < B,$$

and set

$$h(z) = ([(\operatorname{Re} h_1(z)) \vee (-B)] \wedge B) + i([(\operatorname{Im} h_1(z)) \vee (-B)] \wedge B).$$

If we extend h to a bounded continuous function defined in H , and then restrict h to Q , we have the desired function.

(C) Let $d(t)$ be a weakly increasing, positive, real-valued function defined for $0 < t \leq 1$. Then there exists a continuous, complex-valued function k defined in Q , with $|k(z)| \leq 2^{1/2}$ for all $z \in Q$, such that for each $a \in (0, 1]$ and for each arc

$$\gamma \subseteq \{ \langle x, y \rangle : -1 \leq x \leq 1, 0 < y \leq a \},$$

(diameter γ) $\geq d(a)$ implies (diameter $k(\gamma)$) ≥ 2 .

Proof. Let $p(x) = \frac{1}{2} \int_0^x d(t) dt$ ($0 < x \leq 1$). Then p is positive, continuous, and strictly increasing, and $p(x) \leq \frac{1}{2} d(x)$. Let $a \in (0, 1]$ be given. Since $p(x)^{-1}$ is uniformly continuous on each compact subset of $(0, 1]$, there exists $\epsilon \in (0, 1]$ such that

$$(\frac{1}{2}a \leq x_1 \leq 1 \text{ and } |x_1 - x_2| < \epsilon)$$

implies

$$|p(x_1)^{-1} - p(x_2)^{-1}| \leq \frac{1}{2}.$$

Let $\epsilon(a)$ be the supremum of all such ϵ . Then $\epsilon(a)$ is an increasing function of a , and

$$(\frac{1}{2}a \leq x_1 \leq 1 \text{ and } |x_1 - x_2| < \epsilon(a))$$

implies

$$|p(x_1)^{-1} - p(x_2)^{-1}| \leq \frac{1}{2}.$$

Set $q(x) = \int_0^x \epsilon(t) dt$. Then q is positive, continuous, and strictly increasing, and $q(x) \leq \epsilon(x)$. Let $m(x) = \min \{p(x), q(x)\}$. For $\langle x, y \rangle \in Q$, define

$$k_1(y) = \sin(2\pi/ym(y)), \quad k_2(x, y) = \sin(4\pi x/p(y)),$$

$$k(x, y) = k_1(y) + ik_2(x, y).$$

Now suppose that $a \in (0, 1]$ is given, and suppose that $\gamma \subseteq \{ \langle x, y \rangle : -1 \leq x \leq 1, 0 < y \leq a \}$ is an arc with (diameter γ) $\geq d(a)$. Choose $z_1 = \langle x_1, y_1 \rangle$ and $z_2 = \langle x_2, y_2 \rangle$ in γ so that $|z_1 - z_2| \geq d(a)$. Assume without loss of generality that $y_2 \leq y_1$. We can choose a' so that $0 < \frac{1}{2}a' \leq y_1 \leq a' \leq a$. Since $m(a') \leq \frac{1}{2} d(a') \leq \frac{1}{2} d(a)$, and since $|z_1 - z_2| \geq d(a)$, we must have either

$$(12) \quad |y_1 - y_2| \geq m(a')$$

or

$$(13) \quad |y_1 - y_2| < m(a') \text{ and } |x_1 - x_2| \geq \frac{1}{2} d(a').$$

First assume that (12) holds. Here $m(y_2) \leq m(y_1) \leq m(a')$, so

$$2\pi/y_1 m(y_1) \leq 2\pi/y_2 m(y_2),$$

and we have

$$\begin{aligned} \frac{2\pi}{y_2 m(y_2)} - \frac{2\pi}{y_1 m(y_1)} &= \frac{2\pi(y_1 m(y_1) - y_2 m(y_2))}{y_1 y_2 m(y_1) m(y_2)} \\ &\geq \frac{2\pi(y_1 m(y_2) - y_2 m(y_2))}{y_1 y_2 m(y_1) m(y_2)} = \frac{2\pi(y_1 - y_2)}{y_1 y_2 m(y_1)} \\ &\geq \frac{2\pi m(a')}{y_1 y_2 m(y_1)} \geq \frac{2\pi}{y_1 y_2} \geq 2\pi. \end{aligned}$$

Thus, as $\langle x, y \rangle$ moves along γ from $\langle x_1, y_1 \rangle$ to $\langle x_2, y_2 \rangle$, we see that $2\pi/ym(y)$ varies over an interval of length at least 2π , and hence $k_1(y)$ varies over the whole of the interval $[-1, 1]$. Therefore (diameter $k(\gamma)$) ≥ 2 .

Now assume that (13) holds. Then

$$\begin{aligned} \left| \frac{4\pi x_1}{p(y_1)} - \frac{4\pi x_2}{p(y_2)} \right| &\geq 4\pi \left[\left| \frac{x_1}{p(y_1)} - \frac{x_2}{p(y_1)} \right| - \left| \frac{x_2}{p(y_2)} - \frac{x_2}{p(y_1)} \right| \right] \\ &\geq 4\pi \left[\frac{|x_1 - x_2|}{p(y_1)} - \left| \frac{1}{p(y_2)} - \frac{1}{p(y_1)} \right| \right] \\ &\geq 4\pi \left[\frac{\frac{1}{2} d(a')}{p(a')} - \left| \frac{1}{p(y_2)} - \frac{1}{p(y_1)} \right| \right] \\ &\geq 4\pi \left[1 - \left| \frac{1}{p(y_2)} - \frac{1}{p(y_1)} \right| \right]. \end{aligned}$$

Now, $|y_1 - y_2| < m(a') \leq q(a') \leq \epsilon(a')$, so $|p(y_2)^{-1} - p(y_1)^{-1}| \leq \frac{1}{2}$. Therefore

$$|4\pi x_1/p(y_1) - 4\pi x_2/p(y_2)| \geq 2\pi,$$

and we see that as $\langle x, y \rangle$ varies along γ from $\langle x_1, y_1 \rangle$ to $\langle x_2, y_2 \rangle$, the quantity $4\pi x/p(y)$ varies over an interval of length at least 2π , so that $k_2(x, y)$ takes on every value in the interval $[-1, 1]$. Thus (diameter $k(\gamma)$) ≥ 2 .

(D) Let $A \subseteq X$ be a set of type $F_{\sigma\delta}$, and let ϕ be a bounded function of honorary Baire class $\leq 2(A, R^2)$. Then there exists a bounded continuous complex-valued function f defined in Q such that A is the set of curvilinear convergence of f and ϕ is a boundary function for f .

Proof. Let g be the function of (A) and let h be the function of (B). For $t \in (0, 1]$, let

$$\begin{aligned} d_1(t) &= \sup \{ \delta \in (0, 1] : (y_1 \geq t, y_2 \geq t, \langle x_1, y_1 \rangle \in Q, \langle x_2, y_2 \rangle \in Q, \text{ and} \\ &\quad \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle < \delta) \text{ implies } |h(x_1, y_1) - h(x_2, y_2)| \leq t \}, \\ d_2(t) &= \sup \{ \delta \in (0, 1] : (y_1 \geq t, y_2 \geq t, \langle x_1, y_1 \rangle \in Q, \langle x_2, y_2 \rangle \in Q, \text{ and} \\ &\quad \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle < \delta) \text{ implies } |g(x_1, y_1) - g(x_2, y_2)| \leq t \}, \\ d(t) &= \min \{ \frac{1}{3} d_1(\frac{1}{2}t), \frac{1}{3} d_2(\frac{1}{2}t), \frac{1}{3}t \}. \end{aligned}$$

Let k be the function of (C) for this $d(t)$, and set $f(z) = h(z) + g(z)k(z)$ ($z \in Q$). We show that f is the desired function.

Suppose $x \in A$. Then there exists an arc γ at x , lying in $s(x, 1, \frac{3}{8}\pi)$, such that h approaches $\phi(x)$ along γ . But $g(z)$ approaches 0 through $s(x, 1, \frac{3}{8}\pi)$ and k is bounded, so $g(z)k(z)$ approaches 0 along γ . Hence $f(z)$ approaches $\phi(x)$ along γ . Thus ϕ is a boundary function for f , and A is a subset of the set of curvilinear convergence of f . It only remains to show that if x is a point of the set of curvilinear convergence of f , then $x \in A$. To show this, let γ be an arc at x along which f approaches a limit. We may assume without loss of generality that γ has an end point in $\{\langle x, 1 \rangle : -1 \leq x \leq 1\}$. By the properties of g , it will be enough to show that g approaches zero along γ . Assume that g does not approach zero along γ . Then there exists $\epsilon \in (0, 1]$ and there exists a sequence $\{z_n\}$ such that $z_n \in \gamma - \{x\}$, $z_n \rightarrow x$ as $n \rightarrow \infty$, and $|g(z_n)| \geq \epsilon$ for all n . Write $z_n = \langle x_n, y_n \rangle$. Choose N so that $n \geq N$ implies $y_n < \frac{1}{4}\epsilon$.

For the time being, let n be a fixed integer greater than or equal to N . Set $a = 4y_n/3$. Let γ' be the component of $\gamma \cap \text{Cl}[S(d(a), z_n)]$ that contains z_n . (Recall that $S(d(a), z_n) = \{z : |z - z_n| < d(a)\}$.) Then

$$d(a) \leq \text{diameter } \gamma' \leq 2d(a),$$

and, since $d(a) \leq \frac{1}{4}a$,

$$\gamma' \subseteq \{\langle x, y \rangle : \frac{1}{2}a \leq y \leq a\}.$$

By the choice of k , there exist points p and q in γ' with $|k(p) - k(q)| \geq 2$. We have $|p - q| \leq 2d(a) < d_1(\frac{1}{2}a)$, so, by the definition of $d_1(t)$,

$$|h(p) - h(q)| \leq \frac{1}{2}a < \frac{1}{8}\epsilon.$$

Similarly,

$$|g(p) - g(z_n)| \leq \frac{1}{2}a < \frac{1}{8}\epsilon,$$

$$|g(q) - g(z_n)| \leq \frac{1}{2}a < \frac{1}{8}\epsilon.$$

Thus

$$\begin{aligned} |f(p) - f(q)| &\geq |g(p)k(p) - g(q)k(q)| - |h(p) - h(q)| \\ &> |g(p)k(p) - g(z_n)k(p) + g(z_n)k(p) - g(z_n)k(q) \\ &\qquad\qquad\qquad + g(z_n)k(q) - g(q)k(q)| - \frac{1}{8}\epsilon \\ &\geq |g(z_n)| |k(p) - k(q)| - |k(p)| |g(p) - g(z_n)| \\ &\qquad - |k(q)| |g(q) - g(z_n)| - \frac{1}{8}\epsilon \\ &\geq 2\epsilon - 2^{1/2}\frac{1}{8}\epsilon - 2^{1/2}\frac{1}{8}\epsilon - \frac{1}{8}\epsilon > \epsilon. \end{aligned}$$

Note that $|p - z_n| \leq d(a) \leq \frac{1}{4}a = \frac{1}{3}y_n$, and similarly $|q - z_n| \leq \frac{1}{3}y_n$.

We have now shown that, for each $n \geq N$, there exist points $p_n, q_n \in \gamma$ with $|p_n - z_n| \leq \frac{1}{3}y_n$, $|q_n - z_n| \leq \frac{1}{3}y_n$, and $|f(p_n) - f(q_n)| > \epsilon$. But then $p_n \rightarrow x$ and $q_n \rightarrow x$ as $n \rightarrow \infty$, so f does not approach a limit along γ . This is a contradiction. We conclude that $g(z) \rightarrow 0$ along γ , and hence that $x \in A$.

(E) Let $A \subseteq C$ be a set of type $F_{\sigma\delta}$, and let ϕ be a bounded function of honorary Baire class $\leq 2(A, R^2)$. Then there exists a bounded continuous complex-valued function f defined in D such that A is the set of curvilinear convergence of f and ϕ is a boundary function for f .

Proof. If $A = \emptyset$, this is trivial. If $A \neq \emptyset$, then we can assume, by making a suitable rotation of the disk, that $\langle 1, 0 \rangle \in A$. Let $G = D - S(\frac{1}{2}, \langle \frac{1}{2}, 0 \rangle)$ and let $L = C - \{ \langle 1, 0 \rangle \}$. Because $Q \cup X$ is homeomorphic with $G \cup L$, we see from (D) that there exists a bounded continuous complex-valued function f_1 defined in G such that

(i) $A \cap L$ is the set of all points $x \in L$ such that f_1 approaches a limit along some arc at x , and

(ii) the restriction of ϕ to L is a boundary function for f_1 .

Since G is closed relative to D , we can extend f_1 to a bounded continuous function f defined in D in such a way that f has $\phi(\langle 1, 0 \rangle)$ as a radial limit at $\langle 1, 0 \rangle$. This f will have all the desired properties.

(F) Let S^2 denote the Riemann sphere, let $A \subseteq C$ be a set of type $F_{\sigma\delta}$, and let ϕ be a function of honorary Baire class $\leq 2(A, S^2)$. Then there exists a continuous function $f: D \rightarrow S^2$ such that A is the set of curvilinear convergence of f and ϕ is a boundary function for f .

(I) We suppose that

$$S^2 = \{ \langle x, y, z \rangle \in R^3 : x^2 + y^2 + z^2 = 1 \}.$$

We let

$$U = \left\{ \langle x, y, z \rangle \in S^2 : -\frac{1}{2^{1/2}} < z \leq 1 \right\},$$

$$V = \left\{ \langle x, y, z \rangle \in S^2 : -1 \leq z < \frac{1}{2^{1/2}} \right\},$$

$$Z_U = \left\{ \langle x, y, z \rangle \in S^2 : \frac{1}{2^{1/2}} < z \leq 1 \right\},$$

$$Z_V = \left\{ \langle x, y, z \rangle \in S^2 : -1 \leq z < -\frac{1}{2^{1/2}} \right\}.$$

We define mappings $\Phi_U: Z_U \rightarrow U$ and $\Phi_V: Z_V \rightarrow V$ by setting

$$\Phi_U(x, y, z) = \langle x(4z^2 - 1), y(4z^2 - 1), z(4z^2 - 3) \rangle \quad (\langle x, y, z \rangle \in Z_U)$$

and

$$\Phi_V(x, y, z) = \langle x(4z^2 - 1), y(4z^2 - 1), z(4z^2 - 3) \rangle \quad (\langle x, y, z \rangle \in Z_V).$$

Then Φ_U is a one-to-one continuous function from Z_U onto U . Since Z_U and U are each homeomorphic to the unit disk D , it follows from [7, Corollary 1, p. 122] that Φ_U is a homeomorphism of Z_U onto U . Similarly, Φ_V is a homeomorphism of Z_V onto V .

We define a continuous function $\Phi: S^2 \rightarrow S^2$ by setting

$$\Phi(x, y, z) = \Phi_U(x, y, z), \quad \frac{1}{2^{1/2}} < z \leq 1,$$

$$\Phi(x, y, z) = \langle x, y, -z \rangle, \quad -\frac{1}{2^{1/2}} \leq z \leq \frac{1}{2^{1/2}},$$

$$\Phi(x, y, z) = \Phi_V(x, y, z), \quad -1 \leq z < -\frac{1}{2^{1/2}}.$$

Notice that for each $p \in S^2$, the inverse image set $\Phi^{-1}(\{p\})$ contains at most three points.

(II) Most of the results of Hausdorff [2] on real-valued Baire functions can easily be shown to hold also for functions taking values in R^n . We shall make free use of these results in this more general form.

(III) Now we proceed to the proof of (F). Let N be a countable subset of A such that the restriction of ϕ to $A - N$ is of Baire class $\leq 1(A - N, S^2)$, and let $A_1 = A - N$. It will be convenient to let $F_\sigma(A_1)$ denote the class of all subsets of A_1 that are of type F_σ relative to A_1 , and $G_\delta(A_1)$ the class of all subsets of A_1 that are of type G_δ relative to A_1 . Since U and V are open subsets of S^2 and $U \cup V = S^2$, we see that $A_1 \cap \phi^{-1}(U) \in F_\sigma(A_1)$, $A_1 - \phi^{-1}(V) \in G_\delta(A_1)$, and $A_1 - \phi^{-1}(V) \subseteq A_1 \cap \phi^{-1}(U)$. An elegant theorem of Sierpinski [8] now enables us to choose a set $K \in F_\sigma(A_1) \cap G_\delta(A_1)$ such that

$$A_1 - \phi^{-1}(V) \subseteq K \subseteq A_1 \cap \phi^{-1}(U).$$

Let $L = A_1 - K$. Then $L \in F_\sigma(A_1) \cap G_\delta(A_1)$. Moreover, $\phi(K) \subseteq U$ and $\phi(L) \subseteq V$.

Let $p_1 = \langle 1, 0, 0 \rangle$, and define $\psi: A \rightarrow S^2 - \{p_1\}$ as follows. Set

$$\begin{aligned} \psi(x) &= \Phi_U^{-1}(\phi(x)), & x \in K, \\ \psi(x) &= \Phi_V^{-1}(\phi(x)), & x \in L. \end{aligned}$$

If $x \in N$, we let $\psi(x)$ be any element of $Z_U \cup Z_V$ for which $\Phi(\psi(x)) = \phi(x)$. This choice of $\psi(x)$ is always possible, because $\Phi(Z_U \cup Z_V) \supseteq U \cup V = S^2$. Let ψ_0 be the restriction of ψ to $A_1 = K \cup L$. I assert that ψ_0 is of Baire class $\leq 1(A_1, S^2 - \{p_1\})$. Since $S^2 - \{p_1\}$ is homeomorphic to R^2 , it will suffice to show that $\psi_0^{-1}(G) \in F_\sigma(A_1)$ for every open set $G \subseteq S^2 - \{p_1\}$. But

$$\begin{aligned} \psi_0^{-1}(G) &= A_1 \cap \psi^{-1}(G) = [K \cap \psi^{-1}(G)] \cup [L \cap \psi^{-1}(G)] \\ &= [K \cap \phi^{-1}(\Phi_U(Z_U \cap G))] \cup [L \cap \phi^{-1}(\Phi_V(Z_V \cap G))] \in F_\sigma(A_1), \end{aligned}$$

so ψ_0 is of Baire class $\leq 1(A_1, S^2 - \{p_1\})$. Now, $A_1 = A - N$ is of type G_δ relative to A , so (again using the fact that $S^2 - \{p_1\}$ is homeomorphic to R^2) we can extend ψ_0 to a function ψ_1 of Baire class $\leq 1(A, S^2 - \{p_1\})$. The existence of ψ_1 shows that ψ is of honorary Baire class $\leq 2(A, S^2 - \{p_1\})$. The range of ψ is contained in $Z_U \cup Z_V$, so that the values of ψ are bounded away from p_1 . Thus, if we still think of $S^2 - \{p_1\}$ as corresponding to the plane R^2 , ψ corresponds to a bounded function. By (E), there exists a continuous function $f_1: D \rightarrow S^2 - \{p_1\}$ such that the values of f_1 are bounded away from p_1 , A is the set of curvilinear convergence of f_1 , and ψ is a boundary function for f_1 . Let f denote the composite function $\Phi \circ f_1$. Then f is continuous and $\Phi \circ \psi = \phi$ is a boundary function for f . It only remains to show that if x is a point of the set of curvilinear convergence of f , then $x \in A$. Let γ be an arc at x along which f approaches a limit, and let $C(f_1, \gamma)$ denote the cluster set of f_1 along γ . Assume that $x \notin A$. Then f_1 does not approach a limit along γ ,

so $C(f_1, \gamma)$ contains infinitely many points. Now, Φ maps at most three points to any one given point, so $\Phi(C(f_1, \gamma))$ contains infinitely many points. But $\Phi(C(f_1, \gamma))$ is the cluster set of $f = \Phi \circ f_1$ along γ , and hence f does not approach a limit along γ , contrary to our assumption. We conclude that $x \in A$ after all. This completes the proof of the theorem.

The following questions remain open.

PROBLEM 1. *If A is an arbitrary set of type $F_{\sigma\delta}$ in C , does there necessarily exist a continuous real-valued function in D having A as its set of curvilinear convergence?*

PROBLEM 2. *If $A \subseteq C$ is a set of type $F_{\sigma\delta}$, and if ϕ is a function of honorary Baire class $\leq 2(A, R)$, does there necessarily exist a continuous real-valued function in D having A as its set of curvilinear convergence and ϕ as a boundary function?*

Appendix. Some theorems concerning functions of Baire class ≤ 1 which take values on the Riemann sphere can be obtained by the technique used to prove (F). We use the notation set up in the proof of (F).

THEOREM (a). *Let M be a metric space, and let $\phi: M \rightarrow S^2$ be a function such that $\phi^{-1}(G)$ is an F_σ set for every open set $G \subseteq S^2$. Then ϕ is of Baire class $\leq 1(M, S^2)$.*

Proof. Since U and V are open and $U \cup V = S^2$, it follows that the set $\phi^{-1}(U)$ is F_σ , the set $M - \phi^{-1}(V)$ is G_δ , and $M - \phi^{-1}(V) \subseteq \phi^{-1}(U)$. By the theorem of Sierpinski [8], there exists a set K that is simultaneously F_σ and G_δ such that

$$M - \phi^{-1}(V) \subseteq K \subseteq \phi^{-1}(U).$$

Let $L = M - K$. Then L is simultaneously F_σ and G_δ , and

$$\phi(K) \subseteq U, \quad \phi(L) \subseteq V.$$

Define $\psi: M \rightarrow S^2 - \{p_1\}$ (where $p_1 = \langle 1, 0, 0 \rangle$) by setting

$$\psi(x) = \Phi_{\bar{v}}^{-1}(\phi(x)), \quad x \in K, \quad \psi(x) = \Phi_{\bar{v}}^{-1}(\phi(x)), \quad x \in L.$$

If G is an open subset of $S^2 - \{p_1\}$, then

$$\psi^{-1}(G) = [K \cap \phi^{-1}(\Phi_U(Z_U \cap G))] \cup [L \cap \phi^{-1}(\Phi_V(Z_V \cap G))],$$

so $\psi^{-1}(G)$ is an F_σ set. Since $S^2 - \{p_1\}$ is homeomorphic to the plane, it follows that there exists a sequence $\{\psi_n\}$ of continuous functions, each mapping M into $S^2 - \{p_1\}$, such that $\psi_n \rightarrow \psi$ pointwise on M . But then $\Phi(\psi_n(x)) \rightarrow \Phi(\psi(x)) = \phi(x)$ for each fixed $x \in M$, so ϕ is of Baire class $\leq 1(M, S^2)$.

A special case of Theorem (b) was proved (in effect) in [6, proof of Theorem 6] by means of a rather messy lemma (Lemma 3). Theorem (a) provides a proof that is both more general and more esthetically satisfactory.

THEOREM (b). *Let M be a metric space, and let $\phi: M \rightarrow S^2$ be a function. Then ϕ is of honorary Baire class $\leq 2(M, S^2)$ if, and only if, there exists a countable set $N \subseteq M$ such that, for every closed set $F \subseteq S^2$, $\phi^{-1}(F) - N$ is a G_δ set.*

Proof. The implication in one direction is trivial. Now assume that N is countable and that $\phi^{-1}(F) - N$ is a G_δ set for every closed set $F \subseteq S^2$. Let ϕ_0 be the restriction of ϕ to $M - N$. Since S^2 is a subset of R^3 , ϕ_0 is of Baire class $\leq 1(M - N, R^3)$. Because $M - N$ is a G_δ set, ϕ_0 can be extended to a function ϕ_1 of Baire class $\leq 1(M, R^3)$. Now, $\phi_1(x) \in S^2$ except for only countably many x , so there exists some point q in the open ball enclosed by S^2 such that q is not in the range of ϕ_1 . Define a mapping $P: R^3 - \{q\} \rightarrow S^2$ as follows. If $a \in R^3 - \{q\}$, let L be the ray with end point at q which passes through a , and let $P(a)$ be the intersection point of L with S^2 . Then P is continuous and $P(a) = a$ for each $a \in S^2$. Let $\psi = P \circ \phi_1$. If $G \subseteq S^2$ is open, then $\psi^{-1}(G) = \phi_1^{-1}(P^{-1}(G))$, so that $\psi^{-1}(G)$ is an F_σ set. Thus, by Theorem (a), ψ is of Baire class $\leq 1(M, S^2)$. Moreover, if $x \notin N$, then $\phi_1(x) = \phi_0(x) = \phi(x) \in S^2$, so that $\psi(x) = P(\phi(x)) = \phi(x)$. Therefore ϕ is of honorary Baire class $\leq 2(M, S^2)$.

An alternative proof of Theorem (b) could be given by combining Theorem (a) with the following result.

THEOREM (c). *Let M be a metric space, E a G_δ set in M , ϕ a function of Baire class $\leq 1(E, S^2)$. Then ϕ can be extended to a function of Baire class $\leq 1(M, S^2)$.*

To prove this, use the technique appearing in the proof of Theorem (a).

Finally, we note that a theorem proved by Bagemihl and McMillan for real-valued functions [1, Theorem 2] can be transferred to the Riemann sphere by means of our technique.

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