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NOTE ON A PROBLEM OF ALAN SUTCLIFFE

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If n is an integer greater than 1 and $a_h, \cdots, a_1, a_0$ are nonnegative integers, let

$$(a_h, \cdots, a_1, a_0)_n \quad \text{denote} \quad a_h n^h + \cdots + a_1 n + a_0.$$

Thus if $0 \leqq a_i \leqq n-1$ $(i=0, \cdots, h)$, then $a_h, \cdots, a_1, a_0$ are the digits of the number $(a_h, \cdots, a_1, a_0)_n$ relative to the radix n. Alan Sutcliffe studied the problem of finding numbers that are multiplied by an integer when their digits are reversed (*Integers that are multiplied when their digits are reversed*, this MAGAZINE, Similarly,

$$b^2 = -1. \tag{3}$$

Taking q = characteristic of F $(q \cdot 1 = 0)$, choose t and r as specified in the lemma. Using relations (1), (2), (3), we have

$$(t + ra + b)(r^2 + 1 + rta + tb) = r(t^2 + r^2 + 1)a + (t^2 + r^2 + 1)b = 0.$$

One of the factors on the left must be 0, so for some numbers $u, v, w, u \not\equiv 0 \pmod{q}$, we have $w + va + ub = 0$, or $b = -u^{-1}va - u^{-1}w$. So b commutes with a, a contradiction. We conclude that S is not a generalized quaternion group, so S is cyclic.

Thus every Sylow subgroup of F^* is cyclic, and F^* is solvable ([**4**], pp. 181–182). Let Z be the center of F^* and assume $Z \neq F^*$. Then F^*/Z is solvable, and its Sylow subgroups are cyclic. Let A/Z (with $Z \subset A$) be a minimal normal subgroup of F^*/Z. A/Z is an elementary abelian group of order p^k (p prime), so since the Sylow subgroups of F^*/Z are cyclic, A/Z is cyclic. Any group which is cyclic modulo its center is abelian, so A is abelian. Let x be any element of F^*, y any element of A. Since A is normal, $xyx^{-1} \in A$, and $(1+x)y = z(1+x)$ for some $z \in A$. An easy manipulation shows that $y - z = zx - xy = (z - xyx^{-1})x$.

If $y - z = z - xyx^{-1} = 0$, then $y = z = xyx^{-1}$, so x and y commute. Otherwise, $x = (z - xyx^{-1})^{-1}(y - z)$. But A is abelian, and z, y, $xyx^{-1} \in A$, so x commutes with y. Thus we have proven that A is contained in the center of F^*, a contradiction.

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