ON A BOUNDARY PROPERTY OF CONTINUOUS FUNCTIONS

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Let D be the open unit disk in the plane, and let C be its boundary, the unit circle. If x is a point of C, then an *arc at* x is a simple arc γ with one endpoint at x such that $\gamma - \{x\} \subset D$. If f is a function defined in D and taking values in a metric space K, then the *set of curvilinear convergence* of f is

$$\{x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ and there exists a point } p \in K \text{ such that } \lim_{\substack{z \to x \\ z \in \gamma}} f(z) = p\}.$$

J. E. McMillan proved that if f is a continuous function mapping D into the Riemann sphere, then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$ [2, Theorem 5]. In this paper we shall provide a simpler proof of this theorem than McMillan's, and we shall give a generalization and point out some of its corollaries.

Notation. If S is a subset of a topological space, $\overline{S}$ denotes the closure and S^* denotes the interior of S. Of course, when we speak of the interior of a subset of the unit circle, we mean the interior relative to the circle, not relative to the whole plane. Let K be a metric space with metric ρ. If $x_0 \in K$ and $r > 0$, then

$$S(r, x_0) = \{x \in K \mid \rho(x, x_0) < r\}.$$

An arc of C will be called *nondegenerate* if and only if it contains more than one point.

LEMMA 1. *Let* $\mathcal{I}$ *be a family of nondegenerate closed arcs of* C. *Then* $\bigcup_{I \in \mathcal{I}} I - \bigcup_{I \in \mathcal{I}} I^*$ *is countable.*

Proof. Since $\bigcup_{I \in \mathcal{I}} I^*$ is open, we can write $\bigcup_{I \in \mathcal{I}} I^* = \bigcup_n J_n$, where $\{J_n\}$ is a countable family of disjoint open arcs of C. If

$$x_0 \in \bigcup_{I \in \mathcal{I}} I - \bigcup_{I \in \mathcal{I}} I^*,$$

then for some $I_0 \in \mathcal{I}$, x_0 is an endpoint of I_0. For some n, $I_0^* \subset J_n$, so that $x_0 \in \overline{J}_n$. But $x_0 \notin J_n$, so that x_0 is an endpoint of J_n. Thus $\bigcup_{I \in \mathcal{I}} I - \bigcup_{I \in \mathcal{I}} I^*$ is contained in the set of all endpoints of the various J_n; this proves the lemma. ■

In what follows we shall repeatedly use Theorem 11.8 on page 119 in [3] without making explicit reference to it. By a cross-cut we shall always mean a cross-cut of D. Suppose γ is a cross-cut that does not pass through the point 0. If V is the component of $D - \gamma$ that does not contain 0, let $L(\gamma) = \overline{V} \cap C$. Then $L(\gamma)$ is a nondegenerate closed arc of C.

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Suppose Ω is a domain contained in $D - \{0\}$. Let Γ denote the family of all cross-cuts γ with $\gamma \cap D \subset \Omega$. Let

$$I(\Omega) = \bigcup_{\gamma \in \Gamma} L(\gamma), \qquad I_0(\Omega) = \bigcup_{\gamma \in \Gamma} L(\gamma)^*.$$

Let acc(Ω) denote the set of all points on C that are accessible by arcs in Ω.

The following lemma is weaker than it could be, but there is no point in proving more than we need.

LEMMA 2. *The set* acc$(\Omega) - I_0(\Omega)$ *is countable.*

Proof. By Lemma 1, $I(\Omega) - I_0(\Omega)$ is countable; therefore it will suffice to show that acc$(\Omega) - I(\Omega)$ is countable. If acc(Ω) has fewer than two points, we are done. Suppose, on the other hand, that acc(Ω) has two or more points. If $a \in$ acc(Ω), then there exists $a' \in$ acc(Ω) with $a' \neq a$. Let γ, γ' be arcs at a, a', respectively, with

$$\gamma \cap D \subset \Omega, \qquad \gamma' \cap D \subset \Omega.$$

Let p be the endpoint of γ that lies in Ω, p' the endpoint of γ' that lies in Ω. Let $\gamma'' \subset \Omega$ be an arc joining p to p'. The union of γ, γ', and γ'' is an arc δ joining a to a'. By [4], there exists a simple arc $\delta' \subset \delta$ that joins a to a'. Clearly, δ' is a cross-cut with $\delta' \cap D \subset \Omega$ and $a, a' \in L(\delta')$. Thus $a \in I(\Omega)$, and so acc$(\Omega) \subset I(\Omega)$. ∎

LEMMA 3. *Suppose* Ω_1 *and* Ω_2 *are domains contained in* $D - \{0\}$. *If*

$$I_0(\Omega_1) \cap \overline{\text{acc}(\Omega_1)} \quad \textit{and} \quad I_0(\Omega_2) \cap \overline{\text{acc}(\Omega_2)} \tag{1}$$

are not disjoint, then Ω_1 *and* Ω_2 *are not disjoint.*

Proof. We assume Ω_1 and Ω_2 are disjoint, and we derive a contradiction. Let a be a point in both of the two sets (1). Let γ_i be a cross-cut with $\gamma_i \cap D \subset \Omega_i$ such that $a \in L(\gamma_i)^*$ $(i = 1, 2)$. Let U_i and V_i be the components of $D - \gamma_i$, and (to be specific), let U_i be the component containing 0. Note that $\gamma_1 \cap D$ and $\gamma_2 \cap D$ are disjoint.

Suppose $\gamma_1 \cap D \subset V_2$ and $\gamma_2 \cap D \subset V_1$. Then, since $\gamma_1 \cap D \subset \overline{U}_1$, U_1 has a point in common with V_2. But $0 \in U_1 \cap U_2$, so that U_1 has a point in common with U_2 also. Since U_1 is connected, this implies that U_1 has a common point with $\gamma_2 \cap D$, which contradicts the assumption that $\gamma_2 \cap D \subset V_1$. Therefore $\gamma_1 \cap D \not\subset V_2$ or $\gamma_2 \cap D \not\subset V_1$. We conclude that either $\gamma_1 \cap D \subset U_2$ or $\gamma_2 \cap D \subset U_1$. By symmetry, we may assume that $\gamma_2 \cap D \subset U_1$.

It is possible to choose a point $b \in L(\gamma_1)^*$ that is accessible by an arc in Ω_2, because a is in the closure of acc(Ω_2). Let γ be a simple arc joining b to a point of $\gamma_2 \cap D$, such that $\gamma - \{b\} \subset \Omega_2$. Then $\gamma - \{b\}$ and γ_1 are disjoint. Also, $\gamma - \{b\}$ contains a point of U_1 (namely, the point where γ meets $\gamma_2 \cap D$); therefore $\gamma - \{b\} \subset U_1$. Hence $b \in \overline{U}_1$. Since $b \in L(\gamma_1)^*$, this is a contradiction. ∎

THEOREM 1 (J. E. McMillan). *Let* K *be a complete separable metric space, and let* f *be a continuous function mapping* D *into* K. *Let*

$$X = \{x \in C \mid \textit{there exists an arc } \gamma \textit{ at } x \textit{ for which } \lim_{\substack{z \to x \\ z \in \gamma}} f(z) \textit{ exists}\}.$$

Then X *is of type* $F_{\sigma\delta}$.

Proof. Let $\{p_k\}_{k=1}^{\infty}$ be a countable dense subset of K. Let $\{Q(n, m)\}_{m=1}^{\infty}$ be a counting of all sets of the form

$$\left\{re^{it} \,\middle|\, 1 - \frac{1}{n} < r < 1 \text{ and } \theta < t < \theta + \frac{2\pi}{n}\right\},$$

where θ is a rational number. Let $\{U(n, m, k, \ell)\}_{\ell=1}^{\infty}$ be a counting (with repetitions allowed) of the components of

$$f^{-1}\left(S\left(\frac{1}{2^n}, p_k\right)\right) \cap Q(n, m).$$

(We consider $\emptyset$ to be a component of $\emptyset$.) Let

$$A(n, m, k, \ell) = \mathrm{acc}[U(n, m, k, \ell)].$$

Set

$$Y = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}.$$

Since $I_0(U(n, m, k, \ell))$ is open, it is of type F_σ . It follows that Y is of type $F_{\sigma\delta}$.

I claim that $Y \subset X$. Take any $y \in Y$. For each n, choose $m[n]$, $k[n]$, $\ell[n]$ with

(2) $\quad y \in I_0(U(n, m[n], k[n], \ell[n])) \cap \overline{A(n, m[n], k[n], \ell[n])} \qquad (n = 1, 2, 3, \cdots).$

For convenience, set $U_n = U(n, m[n], k[n], \ell[n])$. By (2) and Lemma 3, U_n and U_{n+1} have some point z_n in common. For each n, we can choose an arc $\gamma_n \subset U_{n+1}$ with one endpoint at z_n and the other at z_{n+1} . Then $\gamma_n \subset Q(n+1, m[n+1])$. Also,

$$y \in \overline{A(n+1, m[n+1], k[n+1], \ell[n+1])} \subset \overline{U}_{n+1} \subset \overline{Q(n+1, m[n+1])},$$

and therefore each point of γ_n has distance less than $\frac{2\pi + 1}{n + 1}$ from y. Now $\frac{2\pi + 1}{n + 1} \to 0$ as $n \to \infty$; hence, if we set $\gamma = \{y\} \cup \bigcup_{n=1}^{\infty} \gamma_n$, then γ is an arc with one endpoint at y.

Since U_n and U_{n+1} have a point in common,

$$f^{-1}\left(S\left(\frac{1}{2^n}, p_{k[n]}\right)\right) \quad \text{and} \quad f^{-1}\left(S\left(\frac{1}{2^{n+1}}, p_{k[n+1]}\right)\right)$$

have a common point, and hence

$$S\left(\frac{1}{2^n}, p_{k[n]}\right) \quad \text{and} \quad S\left(\frac{1}{2^{n+1}}, p_{k[n+1]}\right)$$

have a common point. Therefore, if ρ is the metric on K, then

$$\rho(p_{k[n]}, p_{k[n+1]}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} < \frac{1}{2^{n-1}},$$

and therefore

$$\rho(p_{k[n]}, p_{k[n+r]}) \leq \sum_{i=1}^{r} \rho(p_{k[n+i-1]}, p_{k[n+i]}) < \sum_{i=1}^{r} \frac{1}{2^{n+i-2}} < \frac{1}{2^{n-2}}.$$

Thus $\{p_{k[n]}\}$ is a Cauchy sequence and must converge to some point $p \in K$. Because

$$\gamma_n \subset U_{n+1} \subset f^{-1}\left(S\left(\frac{1}{2^{n+1}}, p_{k[n+1]}\right)\right) \quad \text{and} \quad p_{k[n]} \underset{n}{\rightarrow} p,$$

$\lim_{\substack{z\to y \\ z\in\gamma}} f(z) = p$. It is possible that γ is not a simple arc, but by [4] we can replace γ by a simple arc $\gamma' \subset \gamma$. Thus $y \in X$, and we have shown that $Y \subset X$.

Suppose $x \in X$. Let γ_0 be an arc at x such that f approaches a limit p' along γ_0. Take any n. Choose k with $p' \in S\left(\frac{1}{2^n}, p_k\right)$. Choose m so that x is in the interior of $\overline{Q(n, m)} \cap C$. Then γ_0 has a subarc γ_0', with one endpoint at x, such that

$$\gamma_0' - \{x\} \subset Q(n, m) \cap f^{-1}\left(S\left(\frac{1}{2^n}, p_k\right)\right).$$

Hence, for some ℓ, $x \in \operatorname{acc}[U(n, m, k, \ell)] = A(n, m, k, \ell)$. This shows that

$$X \subset \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} A(n, m, k, \ell).$$

By Lemma 2, the set

$$A(n, m, k, \ell) - I_0(U(n, m, k, \ell)) = A(n, m, k, \ell) - [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}]$$

is countable. It follows by a routine argument that

$$\bigcap_{n} \bigcup_{m,k,\ell} A(n, m, k, \ell) - \bigcap_{n} \bigcup_{m,k,\ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}]$$

is countable. Because

$$\bigcap_{n} \bigcup_{m,k,\ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] = Y \subset X \subset \bigcap_{n} \bigcup_{m,k,\ell} A(n, m, k, \ell),$$

the set $X - Y$ is countable, and therefore X is of type $F_{\sigma\delta}$. ■

Before stating our generalization of the foregoing theorem, we must say a few words about spaces of closed sets. If K is a bounded metric space with metric ρ, let $\mathscr{C}(K)$ denote the set of all nonempty closed subsets of K. Hausdorff [1, page 146] defined a metric $\bar{\rho}$ on $\mathscr{C}(K)$ by setting

$$\bar{\rho}(A, B) = \max\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\},$$

where dist(x, E) denotes $\inf_{e \in E} \rho(x, e)$. If K is compact, then $\mathscr{C}(K)$ is a compact metric space with $\bar{\rho}$ as metric [1, page 150].

If f maps D into K and if γ is an arc at a point $x \in C$, we let $C(f, \gamma)$ denote the cluster set of f along γ; that is, we write

$$C(f, \gamma) = \{p \in K \mid \text{there exists a sequence } \{z_n\} \subset \gamma \cap D \text{ such that } z_n \to x \text{ and } f(z_n) \to p\}.$$

THEOREM 2. *Let* K *be a compact metric space, and let* $\mathcal{E}$ *be a closed subset of* $\mathscr{C}(K)$. *Let* $f: D \to K$ *be a continuous function. Then*

$$\{x \in C \mid \textit{there exists an arc } \gamma \textit{ at } x \textit{ and there exists } E \in \mathcal{E} \textit{ such that } C(f, \gamma) \subset E\}$$

is a set of type $F_{\sigma\delta}$.

Proof. If $\varepsilon > 0$ and $E \in \mathscr{C}(K)$, let

$$\mathscr{S}(\varepsilon, E) = \{a \in K \mid \text{there exists } b \in E \text{ with } \rho(a, b) < \varepsilon\}.$$

Note that $\mathscr{S}(\varepsilon, E)$ is open and that

$$F \in \mathscr{C}(K),\ \bar{\rho}(E, F) < \varepsilon \Rightarrow F \subset \mathscr{S}(\varepsilon, E).$$

Let $\{P(k)\}_{k=1}^{\infty}$ be a countable dense subset of $\mathcal{E}$ (such a subset exists, because every compact metric space is separable). Let

$$X = \{x \in C \mid \text{there exist an arc } \gamma \text{ at } x \text{ and an } E \in \mathcal{E} \text{ such that } C(f, \gamma) \subset E\}.$$

Let $\{Q(n, m)\}_{m=1}^{\infty}$ be defined as in the proof of the preceding theorem. Let $\{U(n, m, k, \ell)\}_{\ell=1}^{\infty}$ be a counting (with repetitions allowed) of the components of

$$f^{-1}\left(\mathscr{S}\left(\frac{1}{n}, P(k)\right)\right) \cap Q(n, m).$$

Let $A(n, m, k, \ell) = \operatorname{acc}[U(n, m, k, \ell)]$, and set

$$Y = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}.$$

Since $I_0(U(n, m, k, \ell))$ is open, it is of type F_σ. It follows that Y is of type $F_{\sigma\delta}$.

I claim that $Y \subset X$. Take any $y \in Y$. For each n, choose $m[n], k[n], \ell[n]$ so that

$$y \in I_0(U(n, m[n], k[n], \ell[n])) \cap \overline{A(n, m[n], k[n], \ell[n])}. \tag{3}$$

Set $U_n = U(n, m[n], k[n], \ell[n])$. Since $\mathcal{E}$ is compact, there exist a $P \in \mathcal{E}$ and some strictly ascending sequence $\{n_j\}_{j=1}^{\infty}$ of natural numbers such that

$$P(k[n_j]) \underset{j}{\rightarrow} P.$$

By (3) and Lemma 3, U_{n_j} and $U_{n_{j+1}}$ have some point z_j in common. For each j, choose an arc $\gamma_j \subset U_{n_{j+1}}$ with one endpoint at z_j and the other at z_{j+1}. Then $\gamma_j \subset Q(n_{j+1}, m[n_{j+1}])$. Also,

$$y \in \overline{A(n_{j+1}, m[n_{j+1}], k[n_{j+1}], \ell[n_{j+1}])} \subset \overline{U}_{n_{j+1}} \subset \overline{Q(n_{j+1}, m[n_{j+1}])},$$

and therefore each point of γ_j has distance less than $\frac{2\pi + 1}{n_{j+1}}$ from y. Now $\frac{2\pi + 1}{n_{j+1}} \to 0$ as $j \to \infty$; therefore, if we set $\gamma = \{y\} \cup \bigcup_{j=1}^{\infty} \gamma_j$, then γ is an arc with one endpoint at y.

I claim that $C(f, \gamma) \subset P$. Take any $p \in C(f, \gamma)$. There exists a sequence $\{w_s\}_{s=1}^{\infty}$ in $\gamma - \{y\}$ such that $w_s \underset{s}{\rightarrow} y$ and $f(w_s) \underset{s}{\rightarrow} p$. Let ε be an arbitrary positive number. Choose j_0 so that $\bar{\rho}(P(k[n_j]), P) < \varepsilon/3$ for all $j \geq j_0$. Choose j_1 so that $j \geq j_1$ implies $1/n_{j+1} < \varepsilon/3$. We can choose an s such that $w_s \in \gamma_i$ for some $i \geq j_0, j_1$ and such that

$$\rho(f(w_s), p) < \frac{\varepsilon}{3}. \tag{4}$$

Then

$$f(w_s) \in f(\gamma_i) \subset f(U_{n_{i+1}}) \subset \mathscr{S}\left(\frac{1}{n_{i+1}}, P(k[n_{i+1}])\right),$$

and therefore we can choose a point $q \in P(k[n_{i+1}])$ with

$$\rho(f(w_s), q) < \frac{1}{n_{i+1}} < \frac{\varepsilon}{3}. \tag{5}$$

Moreover, because $\bar{\rho}(P(k[n_{i+1}]), P) < \varepsilon/3$, there exists some $q' \in P$ with

$$\rho(q, q') < \frac{\varepsilon}{3}. \tag{6}$$

Together, (4), (5), and (6) show that $\rho(p, q') < \varepsilon$. Since P is closed and ε is arbitrary, this proves that $p \in P$. Hence $C(f, \gamma) \subset P \in \mathcal{E}$. By [4], we can if necessary replace γ by a simple arc $\gamma' \subset \gamma$; it follows that $y \in X$. Thus $Y \subset X$.

Now suppose $x \in X$. Choose an arc γ_0 at x such that $C(f, \gamma_0) \subset P_0$ for some $P_0 \in \mathcal{E}$. Take any n. Choose k with $\bar{\rho}(P_0, P(k)) < 1/n$. Then

$$P_0 \subset \mathscr{S}\left(\frac{1}{n}, P(k)\right), \quad \text{hence } C(f, \gamma_0) \subset \mathscr{S}\left(\frac{1}{n}, P(k)\right).$$

Choose m so that x is in the interior of $\overline{Q(n, m)} \cap C$.

If for each natural number t there exists a point $z_t' \in \gamma_0 \cap S\left(\frac{1}{t}, x\right) \cap D$ with $z_t' \notin f^{-1}\left(\mathscr{S}\left(\frac{1}{n}, P(k)\right)\right)$, then

$$f(z_t') \in K - \mathscr{S}\left(\frac{1}{n}, P(k)\right),$$

and since $K - \mathscr{S}\left(\frac{1}{n}, P(k)\right)$ is compact, there exist some $a \in K - \mathscr{S}\left(\frac{1}{n}, P(k)\right)$ and a subsequence $\{f(z'_{t_i})\}_{i=1}^{\infty}$ such that $f(z'_{t_i}) \underset{i}{\to} a$. But then $a \in C(f, \gamma_0)$, contrary to the relation $C(f, \gamma_0) \subset \mathscr{S}\left(\frac{1}{n}, P(k)\right)$. We conclude that there exists a natural number t for which

$$\gamma_0 \cap S\left(\frac{1}{t}, x\right) \cap D \subset f^{-1}\left(\mathscr{S}\left(\frac{1}{n}, P(k)\right)\right).$$

It follows that γ_0 has a subarc γ_0' with one endpoint at x such that

$$\gamma_0' - \{x\} \subset f^{-1}\left(\mathscr{S}\left(\frac{1}{n}, P(k)\right)\right) \cap Q(n, m).$$

Hence there exists an ℓ such that

$$x \in \operatorname{acc}[U(n, m, k, \ell)] = A(n, m, k, \ell).$$

This shows that

$$X \subset \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} A(n, m, k, \ell).$$

By Lemma 2, the set

$$A(n, m, k, \ell) - I_0(U(n, m, k, \ell)) = A(n, m, k, \ell) - [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}]$$

is countable. It follows easily that

$$\bigcap_{n} \bigcup_{m,k,\ell} A(n, m, k, \ell) - \bigcap_{n} \bigcup_{m,k,\ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}]$$

is countable. Since

$$\bigcap_{n} \bigcup_{m,k,\ell} [I_0(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] = Y \subset X \subset \bigcap_{n} \bigcup_{m,k,\ell} A(n, m, k, \ell),$$

X - Y must be countable. Thus X is the union of an $F_{\sigma\delta}$-set and a countable set, and hence it is of type $F_{\sigma\delta}$. ∎

In each of the following four corollaries, let f denote a continuous function mapping D into the Riemann sphere.

COROLLARY 1 (J. E. McMillan). *Let* E *be a closed subset of the Riemann sphere. Then the set*

$$\{x \in C \mid \textit{there exist an arc } \gamma \textit{ at } x \textit{ and a point } p \in E \textit{ such that } \lim_{\substack{z \to x \\ z \in \gamma}} f(z) = p\}$$

is of type $F_{\sigma\delta}$.

COROLLARY 2. *Suppose* $d \geq 0$. *Then the set*

$$\{x \in C \mid \textit{there exists an arc } \gamma \textit{ at } x \textit{ such that } [\text{diameter } C(f, \gamma)] \leq d\}$$

is of type $F_{\sigma\delta}$.

COROLLARY 3. *Let* E *be a closed subset of the Riemann sphere. Then the set*

$$\{x \in C \mid \textit{there exists an arc } \gamma \textit{ at } x \textit{ with } C(f, \gamma) \subset E\}$$

is of type $F_{\sigma\delta}$.

COROLLARY 4. *The set*

$$\{x \in C \mid \textit{there exists an arc } \gamma \textit{ at } x \textit{ such that } C(f, \gamma) \textit{ is an arc of a great circle}\}$$

is of type $F_{\sigma\delta}$.

We can obtain all these corollaries by taking $\mathcal{E}$ to be a suitable family of closed sets and applying Theorem 2. To prove Corollary 4, we need the fact that $C(f, \gamma)$ is always connected. One could go on listing such corollaries ad infinitum, but we refrain.

It is interesting to note that in Corollary 1 it is not necessary to assume that E is closed. By combining Corollary 1 with Theorem 6 of [2], one can prove that the conclusion of Corollary 1 holds even if E is merely assumed to be of type G_δ.

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