Boundary Functions for Functions Defined in a Disk[1]

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1. Introduction. Throughout this paper D will denote the open unit disk (in two-dimensional Euclidean space) and C will denote its boundary, the unit circle. Bagemihl and Piranian [2] have introduced the following definition.

Definition. If $x \in C$, an *arc at* x is a simple arc γ having one endpoint at x such that $\gamma - \{x\} \subseteq D$. Let f be any function that is defined in D and takes its values in some metric space S. Then a *boundary function* for f is a function φ on C such that for every $x \in C$ there exists an arc γ at x with

$$\lim_{\substack{z \to x \\ z \in \gamma}} f(z) = \varphi(x).$$

The purpose of this paper is to prove several theorems concerning boundary functions. These theorems include answers to two questions raised in [2] (see Problem 1 and the conjecture on p. 202).

The set of real numbers will be denoted by R, N-dimensional Euclidean space will be denoted by R^N, and the Riemann sphere will be denoted by Σ. Points in R^N will be written in the form $\langle x_1, x_2, \cdots, x_N \rangle$ rather than $(x_1, x_2, \cdots, x_N)$ (to avoid confusion with open intervals of real numbers in the case $N = 2$). Whenever we speak of real-valued functions we mean finite-valued functions, and whenever we speak of increasing functions we refer to weakly increasing (nondecreasing) functions. The abbreviations "l.u.b." and "g.l.b." stand for "least upper bound" and "greatest lower bound" respectively. Finally, it should be noted that our definition of the Baire classes is slightly unconventional (see p. 6 and p.14) in that we consider Baire class α to include Baire class β for every $\beta < \alpha$.

2. Boundary functions for homeomorphisms.

Definition. If $E \subseteq D$, let acc (E) denote the set of all points on C which are accessible by arcs in E.

[1] I would like to thank Professor G. Piranian for his encouragement.

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Lemma 1. *Let A be an arcwise connected subset of D and let B be a connected subset of D. Suppose that $A \cap B = \phi$. Then* acc (A) *and $\bar{B}$ have at most two points in common.*

Proof. Assume that p_1, p_2, p_3 are three distinct points of acc $(A) \cap \bar{B}$ and derive a contradiction. Let γ_i be an arc joining p_i to a point $q_i \in A$, with $\gamma_i - \{p_i\} \subseteq A$ $(i = 1, 2, 3)$. Let γ be an arc in A joining q_1 and q_2. Putting γ_1, γ_2 and γ together, we obtain an arc Γ joining p_1 to p_2, with $\Gamma - \{p_1, p_2\} \subseteq A$. We can assume Γ is a simple arc, for if Γ is not simple, p_1 and p_2 can be joined by some simple arc $\Gamma' \subseteq \Gamma$ (see [7]). Let L_1, L_2 be the two open arcs of C determined by the pair of points p_1, p_2. We may assume, by symmetry, that $p_3 \in L_1$. According to [6] (Theorem 11.8, p. 119), $D - \Gamma$ has two components U_1 and U_2, the boundary of U_1 being $L_1 \cup \Gamma$ and the boundary of U_2 being $L_2 \cup \Gamma$.

Let γ' be an arc in A joining q_3 to a point $q \in \Gamma \cap A$. Putting γ_3 and γ' together, we obtain an arc δ joining p_3 to q. Starting at p_3 and proceeding along δ, let r be the first point of Γ that we reach. Let Δ be the subarc of δ with endpoints at p_3 and r. Clearly, $\Delta - \{p_3\} \subseteq A$. We can assume (according to [7]) that Δ is a simple arc.

Since $p_3 \in L_1$, p_3 is not in $\bar{U}_2$. Since

$$\Delta - \{p_3, r\} \subseteq D - \Gamma = U_1 \cup U_2,$$

$\Delta - \{p_3, r\}$ must have a point in U_1. But $\Delta - \{p_3, r\}$ is connected, so $\Delta - \{p_3, r\} \subseteq U_1$. Hence Δ is a cross cut of U_1. Let M_1, M_2 be the two open subarcs of L_1 with endpoints p_1, p_3 and p_2, p_3 respectively. Let Γ_1, Γ_2 be the two closed subarcs of Γ with endpoints p_1, r and p_2, r respectively. According to [6] (Theorem 11.8, p. 119), $U_1 - \Delta$ has two components V_1 and V_2, the boundary of V_1 being $M_1 \cup \Gamma_1 \cup \Delta$ and the boundary of V_2 being $M_2 \cup \Gamma_2 \cup \Delta$.

Since $\Gamma \cup \Delta \subseteq A$, $B \subseteq V_1 \cup V_2 \cup U_2$. Recall that $p_3 \notin \bar{U}_2$. It follows that since $p_3 \in \bar{B}$, B has a point in common with $V_1 \cup V_2$. But B is connected, so $B \subseteq V_1 \cup V_2$. We see that $p_1 \notin \bar{V}_2$, and therefore that $B \cap V_1 \neq \phi$ (because $p_1 \in \bar{B}$). Hence $B \subseteq V_1$, so $p_2 \in \bar{V}_1$. But, since the boundary of V_1 is $M_1 \cup \Gamma_1 \cup \Delta$, $p_2 \notin \bar{V}_1$. This contradiction proves the lemma.

Lemma 2. *There exists a countable family $\mathcal{S}$ of open disks such that every open set $U \subseteq R^2$ can be written in the form $U = \bigcup_n S_n$, where $S_n \in \mathcal{S}$ and $\bar{S}_n \subseteq U$.*

Proof. Let $\{p_n\}$ be a countable dense subset of R^2, and let $\mathcal{S}$ be the family of all open disks of rational radius having some p_n as center. $\mathcal{S}$ is clearly countable. If U is an open set it is easy to show that for each $x \in U$ there exists an $S_x \in \mathcal{S}$ with $x \in S_x \subseteq \bar{S}_x \subseteq U$. Obviously

$$U = \bigcup_{x \in U} S_x.$$

Theorem 1. *Let f be a homeomorphism of D onto D, and let φ be a boundary function for f. Then there exists a countable set N such that $\varphi|_{C-N}$ is continuous.*

Proof. Take an arbitrary $S \in \mathcal{S}$. It is easily shown that $D \cap S$ and $D - S$ are both connected, so $f^{-1}(D \cap S)$ and $f^{-1}(D - S)$ are both connected. Given $x_0 \in C$, let γ be any arc at x_0. If

$$x_0 \notin \operatorname{acc}(f^{-1}(D \cap S)),$$

then we can choose points on γ arbitrarily close to x_0 which are not in $f^{-1}(D \cap S)$, so

$$x_0 \in \overline{D - f^{-1}(D \cap S)} = \overline{f^{-1}(D - S)}.$$

This shows that

$$C \subseteq \operatorname{acc}(f^{-1}(D \cap S)) \cup \overline{f^{-1}(D - S)}. \tag{1}$$

Let

$$F = \operatorname{acc}(f^{-1}(D \cap S)) \cap \overline{f^{-1}(D - S)}.$$

By Lemma 1, F contains at most two points, and from (1) we see that

$$\operatorname{acc}(f^{-1}(D \cap S)) = F \cup (C - \overline{f^{-1}(D - S)}).$$

Thus we have shown that for each $S \in \mathcal{S}$ we can write

$$\operatorname{acc}(f^{-1}(D \cap S)) = F_S \cup G_S,$$

where F_S is finite and G_S is open (relative to C).

For any arc γ at a point x on C, the *cluster set* $C(f, \gamma)$ of f along γ is defined by

$$C(f, \gamma) = \{w \in R^2 \cup \{\infty\} \mid \text{there exists a sequence } \{z_n\} \subseteq \gamma \cap D \text{ such that } z_n \to x \text{ and } f(z_n) \to w\}.$$

Let

$$E = \{x \in C \mid \text{there exist arcs } \gamma_1, \gamma_2 \text{ at } x \text{ such that } C(f, \gamma_1) \cap C(f, \gamma_2) = \phi\}.$$

A theorem of Bagemihl [1] states that E is countable. Let

$$N = E \cup \bigcup_{S \in \mathcal{S}} F_S .$$

N is countable. Let φ_0 denote the restriction of φ to $C - N$.

If U is any open set, write $U = \bigcup_n S_n$, where $S_n \in \mathcal{S}$, $\bar{S}_n \subseteq U$. Suppose $x \in \varphi_0^{-1}(U)$. Then $\varphi_0(x) = \varphi(x) \in S_n$ for some n, which implies that $x \in \operatorname{acc}(f^{-1}(S_n \cap D))$. Thus

$$\varphi_0^{-1}(U) \subseteq \bigcup_n \operatorname{acc}(f^{-1}(S_n \cap D)) - N.$$

On the other hand, suppose $x \in \operatorname{acc}(f^{-1}(S_n \cap D))$ for some n, and $x \notin N$. Choose an arc γ in $f^{-1}(S_n \cap D)$ with one endpoint at x. Clearly,

$$C(f, \gamma) \subseteq \overline{S_n \cap D} \subseteq \bar{S}_n \subseteq U.$$

Since $x \notin E$,

$$\varphi_0(x) = \varphi(x) \in C(f, \gamma) \subseteq U,$$

so $x \in \varphi_0^{-1}(U)$. Thus

$$\bigcup_n \operatorname{acc}(f^{-1}(S_n \cap D)) - N \subseteq \varphi_0^{-1}(U),$$

so

$$\varphi_0^{-1}(U) = \bigcup_n \operatorname{acc}(f^{-1}(S_n \cap D)) - N = \bigcup_n (F_{S_n} \cup G_{S_n}) - N$$
$$= \bigcup_n G_{S_n} - N = (\bigcup_n G_{S_n}) \cap (C - N).$$

Thus, for each open set U, $\varphi_0^{-1}(U)$ is an open set relative to C–N. Therefore φ_0 is continuous. *Q.E.D.*

3. Boundary functions for continuous functions.

Definition. Let S and T be metric spaces. We will say the function f is of *Baire class* $1(S, T)$ *if*, and only if,

(i) domain $f = S$,
(ii) range $f \subseteq T$, and
(iii) there exists a sequence $\{f_n\}$ of continuous functions, each mapping S into T, such that $f_n \to f$ pointwise on S.

We will say the function g is of *honorary Baire class* $2(S, T)$ if, and only if,
(i) domain $g = S$,
(ii) range $g \subseteq T$, and
(iii) there exists a function f of Baire class $1(S, T)$ and a countable set N such that $f|_{S-N} = g|_{S-N}$.

Lemma 3. *Let f be a continuous real-valued function in D and let φ be a finite-valued boundary function for f. Let r and t be real numbers with $r < t$. Then*

(A) *there exists a G_δ set G and a countable set N such that*

$$\varphi^{-1}([r, +\infty)) \supseteq G \supseteq \varphi^{-1}([t, +\infty)) - N, \textit{ and}$$

(B) *there exists a G_δ set H and a countable set M such that*

$$\varphi^{-1}((-\infty, t]) \supseteq H \supseteq \varphi^{-1}((-\infty, r]) - M.$$

Proof. Let

$$\epsilon = \frac{t - r}{2},$$

$$C_n = \left\{z \in R^2 \mid |z| = 1 - \frac{1}{n}\right\},$$

$$A_n = \left\{z \in R^2 \mid 1 - \frac{1}{n} < |z| < 1\right\},$$

$$E_n = \{x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ having one endpoint on } C_n, \text{ with } \gamma - \{x\} \subseteq f^{-1}((-\infty, r))\},$$

$$K = \{x \in C \mid \text{there exists an arc } \gamma \text{ at } x \text{ with } \gamma - \{x\} \subseteq f^{-1}((t - \epsilon, +\infty))\}.$$

Observe that

$$\varphi^{-1}((-\infty, r)) \subseteq \bigcup_{n=1}^{\infty} E_n,$$

and

$$\varphi^{-1}((t - \epsilon, +\infty)) \subseteq K.$$

For the time being, let n be a fixed integer. If $x \in K$, we can find an arc γ_x at x such that

$$\gamma_x - \{x\} \subseteq A_n \cap f^{-1}([t - \epsilon, +\infty)).$$

Since an arc at x is by definition a simple arc, $\gamma_x - \{x\}$ is a connected set. It follows that $\gamma_x - \{x\}$ must be contained entirely within one component of the open set

$$A_n \cap f^{-1}((t - \epsilon, +\infty)).$$

We denote this component by U_x. U_x is a nonempty open connected set.

Let T be the set of all points of K which are two-sided limit points of $\bar{E}_n$.

Assertion. If $x, y \in T$ and $x \neq y$, then $U_x \cap U_y = \phi$.

To prove this assertion we assume that z is a point of $U_x \cap U_y$ and we derive a contradiction. Choose points x' and y' in $\gamma_x - \{x\}$ and $\gamma_y - \{y\}$ respectively. Join x to x' by an appropriate subarc of γ_x. Join x' to z by an arc in U_x. Join z to y' by an arc in U_y. Join y' to y by a subarc of γ_y. Putting these arcs together, we obtain an arc α with endpoints at x and y such that

$$\alpha - \{x, y\} \subseteq A_n \cap f^{-1}((t - \epsilon, +\infty)).$$

We can assume that α is a simple arc, for if α is not a simple arc we can replace α by a simple arc $\alpha' \subseteq \alpha$ having endpoints at x and y (see [7]). α is a crosscut of D. Let L_1 and L_2 be the two open arcs of C determined by x and y. According to [6] (Theorem 11.8, p. 119), $D - \alpha$ has two components, V_1 and V_2, whose boundaries are $L_1 \cup \alpha$ and $L_2 \cup \alpha$ respectively. From the fact that C_n is connected and does not intersect α it follows that C_n is contained entirely within one component of $D - \alpha$. By symmetry, we may assume $C_n \subseteq V_2$.

Since x is a two-sided limit point of $\bar{E}_n$, L_1 must contain a point of $\bar{E}_n$, and hence a point of E_n. Say $w \in L_1 \cap E_n$. There exists a simple arc β joining w to some point on C_n, with

$$\beta - \{w\} \subseteq f^{-1}((-\infty, r)).$$

$\beta - \{w\}$ cannot have a point in common with α, because

$$\alpha - \{x, y\} \subseteq f^{-1}((t - \epsilon, +\infty)),$$

and

$$f^{-1}((-\infty, r)) \cap f^{-1}((t - \epsilon, +\infty)) = \phi.$$

Thus $C_n \cup (\beta - \{w\})$ is a connected set not meeting α. $C_n \cup (\beta - \{w\})$ meets V_2, so $C_n \cup (\beta - \{w\}) \subseteq V_2$. Consequently, w is in the boundary of V_2. But this is a contradiction, because $w \varepsilon L_1$ and the boundary of V_2 is $L_2 \cup \alpha$. This proves the assertion.

From the assertion it follows immediately that T is countable; for any family of disjoint nonempty open sets is countable. We know that the set S of all points of $\bar{E}_n$ which are not two-sided limit points of $\bar{E}_n$ is countable.

$$K \cap \bar{E}_n = [K \cap S] \cup [K \cap (\bar{E}_n - S)] = (K \cap S) \cup T.$$

This shows that (for any n) $K \cap \bar{E}_n$ is countable. So if we let

$$N = K \cap \bigcup_{n=1}^{\infty} \bar{E}_n = \bigcup_{n=1}^{\infty} (K \cap \bar{E}_n),$$

then N is a countable set. Let

$$G = C - \bigcup_{n=1}^{\infty} \bar{E}_n .$$

G is a G_δ set. Using the fact that

$$\varphi^{-1}((-\infty, r)) \subseteq \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bar{E}_n ,$$

we find that

$$C - \varphi^{-1}((-\infty, r)) \supseteq C - \bigcup_{n=1}^{\infty} \bar{E}_n = G \supseteq K - N.$$

But

$$C - \varphi^{-1}((-\infty, r)) = \varphi^{-1}([r, +\infty))$$

and

$$K \supseteq \varphi^{-1}((t - \epsilon, +\infty)) \supseteq \varphi^{-1}([t, +\infty)),$$

so

$$\varphi^{-1}([r, +\infty)) \supseteq G \supseteq K - N \supseteq \varphi^{-1}([t, +\infty)) - N.$$

This proves (A). To prove (B), simply replace f and φ by $-f$ and $-\varphi$, and apply (A).

Theorem 2. *Let f be a continuous real-valued function in D, and let φ be a finite-valued boundary function for f. Then φ is of honorary Baire class $2(C, R)$.*

Proof. For each pair of rational numbers r and t with $r < t$, choose G_δ

sets $G(r, t)$, $H(r, t)$ and countable sets $N(r, t)$, $M(r, t)$ such that

$$\varphi^{-1}([r, +\infty)) \supseteq G(r, t) \supseteq \varphi^{-1}([t, +\infty)) - N(r, t),$$

and

$$\varphi^{-1}((-\infty, t]) \supseteq H(r, t) \supseteq \varphi^{-1}((-\infty, r]) - M(r, t).$$

Let

$$N = \bigcup_{r,t} [N(r, t) \cup M(r, t)],$$

where the union is taken over all pairs of rationals r, t with $r < t$. N is countable. Let φ_0 denote the restriction of φ to $C - N$, and let $G^*(r, t) = G(r, t) - N$. Since every countable set is an F_σ set, $G^*(r, t)$ is a G_δ set. Observe that

$$\text{(2)} \qquad \begin{aligned} \varphi_0^{-1}([r, +\infty)) &= \varphi^{-1}([r, +\infty)) - N \supseteq G^*(r, t) \\ &\supseteq \varphi^{-1}([t, +\infty)) - N = \varphi_0^{-1}([t, +\infty)). \end{aligned}$$

If t is a fixed rational number, let $\{r_n\}$ be a strictly increasing sequence of rational numbers converging to t. Then, by (2),

$$\bigcap_{n=1}^{\infty} \varphi_0^{-1}([r_n, +\infty)) \supseteq \bigcap_{n=1}^{\infty} G^*(r_n, t) \supseteq \varphi_0^{-1}([t, +\infty)) = \bigcap_{n=1}^{\infty} \varphi_0^{-1}([r_n, +\infty)),$$

so

$$\varphi_0^{-1}([t, +\infty)) = \bigcap_{n=1}^{\infty} G^*(r_n, t).$$

This proves that for every rational t, $\varphi_0^{-1}([t, +\infty))$ is a G_δ set.

If u is any real number, choose a strictly increasing sequence $\{t_n\}$ of rational numbers converging to u. Then

$$\varphi_0^{-1}([u, +\infty)) = \bigcap_{n=1}^{\infty} \varphi_0^{-1}([t_n, +\infty)),$$

so $\varphi_0^{-1}([u, +\infty))$ is a G_δ set. By a similar argument, we find that $\varphi_0^{-1}((-\infty, u])$ is a G_δ set for every real u. So

$$\varphi_0^{-1}((u, +\infty)) = (C - N) \cap (C - \varphi_0^{-1}((-\infty, u]))$$

is the intersection of an F_σ set with $C - N$. By a theorem stated on p. 309 of Hausdorff's paper [5], φ_0 can be extended to a real-valued function φ_1 on C such that for every real u, $\varphi_1^{-1}([u, +\infty))$ is a G_δ set and $\varphi_1^{-1}((u, +\infty))$ is an F_σ set. By Theorem IX of the same paper, φ_1 is of Baire class $1(C, R)$. Since $\varphi(x) = \varphi_1(x)$ except for $x \varepsilon N$, φ is of honorary Baire class $2(C, R)$. *Q.E.D.*

Corollary. *Let f be a continuous function mapping D into R^N, and suppose $\varphi : C \to R^N$ is a boundary function for f. Then φ is of honorary Baire class $2(C, R^N)$.*

Proof. We simply write our functions in terms of their components, say

$$f = \langle f_1, f_2, \cdots, f_N \rangle, \quad \text{and} \quad \varphi = \langle \varphi_1, \varphi_2, \cdots, \varphi_N \rangle.$$

Obviously φ_i is a boundary function for f_i, and so is of honorary Baire class $2(C, R)$. We choose a function g_i of Baire class $1(C, R)$ which agrees with φ_i except on a countable set M_i. Setting

$$g = \langle g_1, g_2, \cdots, g_N \rangle,$$

it is clear that g is of Baire class $1(C, R^N)$, and that g agrees with φ except on the countable set $\bigcup_{i=1}^{N} M_i$. Hence φ is of honorary Baire class $2(C, R^N)$.

Q.E.D.

Lemma 4. *Let g be a continuous function mapping C into R^3. Let q be a point of R^3 and let ϵ be a positive real number. Then there exists a continuous function $g^* : C \to R^3$ such that q does not lie in the range of g^*, and for all $x \in C$,*

$$|g(x) - q| \geqq \epsilon \Rightarrow g(x) = g^*(x).$$

Proof. Let

$$S = \{y \in R^3 \mid |y - q| < \epsilon\}.$$

If $g(C) \subseteq S$, let $g^* : C \to R^3$ be any continuous function whose range does not include q. Otherwise, $g^{-1}(S)$ is a proper open subset of C and hence can be written in the form

$$g^{-1}(S) = \bigcup_k I_k,$$

where

$$I_k = \{e^{it} \mid a_k < t < b_k\},$$

and

$$k \neq l \Rightarrow I_k \cap I_l = \phi.$$

Since $g^{-1}(\{q\})$ is a closed (and therefore compact) subset of $g^{-1}(S)$, $g^{-1}(\{q\})$ is covered by a finite number of I_k's. Say

$$g^{-1}(\{q\}) \subseteq I_1 \cup I_2 \cup \cdots \cup I_n.$$

The endpoints e^{ia_k} and e^{ib_k} of I_k are not in $g^{-1}(\{q\})$, so we can construct, for each k, a continuous function $g_k : \bar{I}_k \to R^3$ such that

$$g_k(e^{ia_k}) = g(e^{ia_k}), \quad g_k(e^{ib_k}) = g(e^{ib_k}),$$

and q is not in the range of g_k. Define

$$g^*(x) = g(x), \quad \text{if} \quad x \in C - (I_1 \cup I_2 \cup \cdots \cup I_n),$$

$$g^*(x) = g_k(x), \quad \text{if} \quad x \in I_k, \qquad k = 1, \cdots, n.$$

It is easy to show that g^* has the desired properties.

Theorem 3. *Let f be a continuous function mapping D into the Riemann* *sphere Σ, and let φ be a boundary function for f. Then φ is of honorary Baire class* $2(C, \Sigma)$.

Proof. Since Σ is a subset of R^3, the corollary to Theorem 2 shows that φ is of honorary Baire class $2(C, R^3)$. Let g be a function of Baire class $1(C, R^3)$ which differs from φ only on a countable set N. Then $g(C) - \Sigma$ is countable, so there exists a point q inside of Σ (that is, in the bounded open domain determined by Σ) which is not in the range of g. Let $\{g_n\}$ be a sequence of continuous functions converging to g. By Lemma 4 we can find (for each n) a continuous function $g_n^* : C \to R^3$ such that q does not lie in the range of g_n^*, and for all $x \varepsilon C$,

$$|g_n(x) - q| \geqq \frac{1}{n} \Rightarrow g_n(x) = g_n^*(x).$$

It is easy to show that $g_n^* \to g$.

We define a function P as follows. If $a \varepsilon R^3 - \{q\}$, let l be the unique ray with endpoint at q that passes through a, and let $P(a)$ be the intersection point of l with Σ. Obviously, P is a continuous mapping of $R^3 - \{q\}$ onto Σ, and P fixes every point of Σ. Therefore

$$P(g(x)) = \varphi(x), \qquad \text{if} \quad x \notin N,$$

$P(g_n^*(x))$ is a continuous function from C into Σ, and

$$P(g_n^*(x)) \to P(g(x)) \quad \text{as} \quad n \to \infty.$$

This shows that φ is of honorary Baire class $2(C, \Sigma)$. *Q.E.D.*

4. Boundary functions for Baire functions. In this section we concern ourselves only with real-valued functions. We shall prove that a boundary function for a function of Baire class $\alpha \geqq 1$ is of Baire class $\alpha + 1$. It is convenient to prove this theorem for functions that are defined in the (open) upper half-plane and have boundary functions defined on the x-axis rather than for functions defined in D. Once the theorem is proved in this form it is a routine computational matter to show that it also holds for functions defined in D. The reader should be familiar with the results of Hausdorff [5] before reading this section. Unfortunately, we must begin with some tedious preliminaries.

Let

$$D^0 = \{\langle x, y\rangle \mid x, y \varepsilon R, y > 0\},$$

$$C^0 = \{\langle x, 0\rangle \mid x \varepsilon R\},$$

$$C_n^0 = \left\{\left\langle x, \frac{1}{n}\right\rangle \,\middle|\, x \varepsilon R\right\},$$

$$A_n^0 = \left\{\langle x, y\rangle \mid x, y \varepsilon R, 0 < y < \frac{1}{n}\right\}.$$

We will regard C^0 as being identical with R.

Suppose S is a metric space. Let $\mathcal{G}_S$ be the class of all open sets of S and let $\mathfrak{F}_S$ be the class of all closed sets of S.

A function $f : S \to R$ is of *Baire class* 0 if and only if it is continuous. For any ordinal number $\alpha > 0$, f is of *Baire class* α if and only if f is the pointwise limit of a sequence of functions each of Baire class less than α.

Let $\mathfrak{M}_S^\alpha$ denote the class of all sets $M \subseteq S$ such that

$$M = f^{-1}((r, +\infty)),$$

for some real r and some function f of Baire class α on S. Let $\mathfrak{N}_S^\alpha$ denote the class of all sets $N \subseteq S$ such that

$$N = f^{-1}([r, +\infty)),$$

for some real r and some function f of Baire class α on S. It is easily shown that $\mathfrak{M}_S^0 = \mathcal{G}_S$ and $\mathfrak{N}_S^0 = \mathfrak{F}_S$.

Let

$$\mathcal{G} = \mathcal{G}_{C^\circ} = \mathcal{G}_R ,$$

$$\mathfrak{F} = \mathfrak{F}_{C^\circ} = \mathfrak{F}_R ,$$

$$\mathfrak{M}^\alpha = \mathfrak{M}_{C^\circ}^\alpha = \mathfrak{M}_R^\alpha ,$$

$$\mathfrak{N}^\alpha = \mathfrak{N}_{C^\circ}^\alpha = \mathfrak{N}_R^\alpha ,$$

If $\mathfrak{O}$ is any class of sets, let $\mathfrak{O}_\sigma$ denote the class of all countable unions of members of $\mathfrak{O}$, and let $\mathfrak{O}_\delta$ denote the class of all countable intersections of members of $\mathfrak{O}$.

Each of the following facts is either explicitly stated in [5], or can be easily deduced from statements found in [5], or is obtained by a routine transfinite induction argument.

I. $$\mathfrak{M}_S^\alpha = \Big(\bigcup_{\lambda<\alpha} \mathfrak{N}_S^\lambda\Big)_{\delta\sigma} , \qquad \mathfrak{N}_S^\alpha = \Big(\bigcup_{\lambda<\alpha} \mathfrak{M}_S^\lambda\Big)_{\sigma\delta} .$$

II. Let A be any subset of the metric space S. If f is a function of Baire class α on S, then $f|_A$ is a function of Baire class α on A.

III. Let f be a function of Baire class α whose domain contains $\{\langle x, b\rangle \mid x \in R\}$. Then $f(\langle x, b\rangle)$ is a function (of x) of Baire class α.

IV. If $A \subseteq S$, then

$$\mathfrak{M}_A^\alpha = \{M \cap A \mid M \in \mathfrak{M}_S^\alpha\},$$

$$\mathfrak{N}_A^\alpha = \{N \cap A \mid N \in \mathfrak{N}_S^\alpha\}.$$

V. If f is of Baire class α on S, then for each real r,

$$f^{-1}((-\infty, r)) \in \mathfrak{M}_S^\alpha ,$$

and

$$f^{-1}((-\infty, r]) \in \mathfrak{N}_S^\alpha .$$

VI. If $\alpha \geqq 2$, then $(\mathcal{G}_S)_\delta \cup (\mathfrak{F}_S)_\sigma \subseteq \mathfrak{M}_S^\alpha \cap \mathfrak{N}_S^\alpha$.

VII. $E \in \mathfrak{N}_S^\alpha \Leftrightarrow S - E \in \mathfrak{M}_S^\alpha$.

VIII. $\mathfrak{M}_S^\alpha$ and $\mathfrak{N}_S^\alpha$ are closed under finite unions and intersections. $\mathfrak{M}_S^\alpha$ is closed under countable unions and $\mathfrak{N}_S^\alpha$ is closed under countable intersections.

IX. Let f be a real-valued function on S. Suppose that for every real r

$$f^{-1}([r, +\infty)) \in \mathfrak{N}_S^\alpha ,$$

and

$$f^{-1}((r, +\infty)) \in \mathfrak{M}_S^\alpha .$$

Then f is of Baire class α.

Definition. If A and B are two sets, we will call A and B *equivalent*, and write $A \sim B$, if and only if $A - B$ and $B - A$ are both countable. It is easily verified that $\sim$ is an equivalence relation.

Lemma 5. *If $A \sim E$, then $S - A \sim S - E$ for any set S. If $A_n \sim E_n$ (for all n in some countable set N), then*

$$\bigcup_{n \in N} A_n \sim \bigcup_{n \in N} E_n \quad \textit{and} \quad \bigcap_{n \in N} A_n \sim \bigcap_{n \in N} E_n .$$

The proof of this lemma is routine.

Definition. An interval of real numbers will be called *nondegenerate* if it contains more than one point.

Lemma 6. *Any union of nondegenerate intervals is equivalent to an open set.*

Proof. Let $\mathcal{I}$ be a family of nondegenerate intervals and let $H = \cup \mathcal{I}$. For any x and y let

$$I(x, y) = [x, y], \quad \text{if} \quad x \leqq y,$$

and let

$$I(x, y) = [y, x], \quad \text{if} \quad y \leqq x.$$

Define a relation $\mathfrak{R}$ on H by

$$x \mathfrak{R} y \Leftrightarrow I(x, y) \subseteq H, \qquad (x, y \in H).$$

It is easy to show that $\mathfrak{R}$ is an equivalence relation on H. In view of the fact that a set A of real numbers is an interval if and only if

$$x, y \in A \Rightarrow I(x, y) \subseteq A,$$

it is obvious that each equivalence class is an interval. For each $x \in H$, there exists an $I \in \mathcal{I}$ with $x \in I$. Every member of I is equivalent to x. Thus each equivalence class contains more than one point, and hence is a nondegenerate interval. Let $\{J_\alpha\}$ be the family of equivalence classes. Any disjoint family of nondegenerate intervals is countable, so there are only countably many J_α's. Let E be the set of all endpoints of the various J_α's. Then E is countable and

$$H = \bigcup_\alpha J_\alpha \sim \bigcup_\alpha J_\alpha - E = \bigcup_\alpha J^*_\alpha\,,$$

where J^*_α is the interior of J_α. This proves the lemma.

Lemma 7. *Let h be an increasing real-valued function on a nonempty set $E \subseteq R$. Suppose that $|x - h(x)| \leqq 1$ for every $x \varepsilon E$. Then h can be extended to an increasing real-valued function h_1 on R.*

Proof. Let $e = \text{g.l.b. } E$ (e may be $-\infty$). For each $x_0 \varepsilon (e, +\infty)$ set

$$h_1(x_0) = \text{l.u.b. } \{h(x) \mid x \varepsilon (-\infty, x_0] \cap E\}.$$

Since $|x - h(x)| \leqq 1$ for all $x \varepsilon E$,

$$x \varepsilon (-\infty, x_0] \cap E \Rightarrow h(x) \leqq x_0 + 1,$$

so h_1 is finite-valued. If $e = -\infty$ we are done. If $e > -\infty$, then $x \varepsilon E$ implies $h(x) \geqq e - 1$, so h is bounded below. For $x_0 \varepsilon (-\infty, e]$ set

$$h_1(x_0) = \text{g.l.b. } \{h(x) \mid x \varepsilon E\}.$$

It is easily verified that h_1 has the desired properties.

Lemma 8. *Let f be a real-valued function of Baire class α on R. Let h be an increasing real-valued function on R. Set $g(x) = f(h(x))$. Then there exists a countable set N such that $g\,|_{R-N}$ is of Baire class α.*

Proof. It is well known that an increasing function has at most countably many discontinuities. Let M be the set of discontinuity points of h. If f is of Baire class 0, then g is continuous at all points of $R - M$, so $g\,|_{R-M}$ is of Baire class 0. This proves the lemma for the case $\alpha = 0$.

We now proceed by transfinite induction. Suppose the lemma holds for every ordinal $\lambda < \alpha$. If f is of Baire class α we may choose a sequence of functions $\{f_n\}$ converging to f, where f_n is of Baire class $\lambda_n < \alpha$. If we set $g_n(x) = f_n(h(x))$ it is clear that $g_n(x) \to f(h(x)) = g(x)$. By the induction hypothesis we may choose (for each n) a countable set N_n such that $g_n\,|_{R-N_n}$ is of Baire class λ_n. Let $N = \bigcup_{n=1}^\infty N_n$. Then $g_n\,|_{R-N}$ is of Baire class λ_n, and since $g_n\,|_{R-N} \to g\,|_{R-N}$, $g\,|_{R-N}$ is of Baire class α. This proves the lemma.

Theorem 4. *Let f be a real-valued function of Baire class $\alpha \geqq 1$ on D^0, and let φ be a finite-valued boundary function for f. Then φ is of Baire class $\alpha + 1$.*

Proof. Let r and t be two real numbers with $r < t$. r and t will remain fixed throughout the first part of the proof. Set

$$P = \varphi^{-1}((-\infty, r]),$$
$$Q = \varphi^{-1}([t, +\infty)),$$
$$E = P \cup Q,$$
$$\epsilon = \frac{t - r}{4}.$$

Observe that $P \cap Q = \phi$. For each $x \varepsilon E$, choose an arc γ_x at x such that

$$\lim_{\substack{z \to x \\ z \varepsilon \gamma_x}} f(z) = \varphi(x), \qquad \gamma_x \subseteq \{z \mid |z - x| \leqq 1\},$$

and

$$(a) \quad f(\gamma_x) \subseteq (-\infty, r + \epsilon), \quad \text{if} \quad x \varepsilon P$$

$$(b) \quad f(\gamma_x) \subseteq (t - \epsilon, +\infty), \quad \text{if} \quad x \varepsilon Q.$$

(This is accomplished by cutting the arc off sufficiently close to x.) We remark that if $x \varepsilon P$ and $y \varepsilon Q$, then $\gamma_x \cap \gamma_y = \phi$.

We will say that γ_x meets γ_y in $\bar{A}_n^0$ provided that γ_x and γ_y have subarcs γ_x' and γ_y' respectively such that $x \varepsilon \gamma_x' \subseteq \bar{A}_n^0$, $y \varepsilon \gamma_y' \subseteq \bar{A}_n^0$, and $\gamma_x' \cap \gamma_y' \neq \phi$. Let

$$\begin{aligned} L_0 &= \{x \varepsilon P \mid (\forall n)(\exists y \neq x)(\gamma_x \text{ meets } \gamma_y \text{ in } \bar{A}_n^0)\}, \\ L_1 &= \{x \varepsilon Q \mid (\forall n)(\exists y \neq x)(\gamma_x \text{ meets } \gamma_y \text{ in } \bar{A}_n^0)\}, \\ M_0 &= \{x \varepsilon P \mid (\exists n)(\gamma_x \text{ meets no } \gamma_y \ (y \neq x) \text{ in } \bar{A}_n^0)\}, \\ M_1 &= \{x \varepsilon Q \mid (\exists n)(\gamma_x \text{ meets no } \gamma_y \ (y \neq x) \text{ in } \bar{A}_n^0)\}, \\ L &= L_0 \cup L_1, \\ M &= M_0 \cup M_1. \end{aligned}$$

Observe that L_0, L_1, M_0, M_1 are pairwise disjoint, and that $P = L_0 \cup M_0$ and $Q = L_1 \cup M_1$.

For each $x \varepsilon M$, let n_x be an integer such that γ_x meets no γ_y (with $y \neq x$) in $\bar{A}_{n_x}^0$. Notice that $n \geqq n_x$ implies γ_x meets no γ_y in $\bar{A}_n^0$. Let

$$K_n = \{x \varepsilon E \mid \gamma_x \text{ meets } C_n^0, \text{ and if } x \varepsilon M,\ n_x \leqq n\}.$$

Clearly $E = \bigcup_{n=1}^{\infty} K_n$. Moreover, $K_n \subseteq K_{n+1}$ for each n.

Take any fixed integer n. For each $x \varepsilon L_0$ we can find a $y \neq x$ such that γ_x meets γ_y in $\bar{A}_n^0$. Let I_x^n be the nondegenerate closed interval between x and y. We shall show that $I_x^n \subseteq L_0 \cup (C^0 - K_n)$. If $t \varepsilon I_x^n$, either $t \varepsilon C^0 - K_n$ or $t \varepsilon K_n$. Suppose $t \varepsilon K_n$. Then γ_t meets C_n^0, and (if $t \varepsilon M$) $n_t \leqq n$. It is clear from Figure 1 that γ_t must meet either γ_x or γ_y in $\bar{A}_n^0$. (This can be rigorized by means of Theorem 11.8 on p. 119 in [6].)

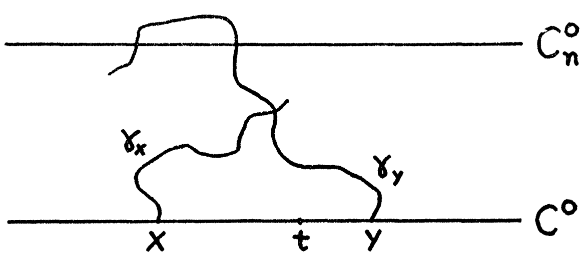

FIGURE 1.

Consequently, $t \notin M$. Now $x \varepsilon L_0 \subseteq P$, so since γ_x intersects γ_y, $y \notin Q$. So $y \varepsilon E - Q = P$. Similarly, since γ_t meets γ_x or γ_y, $t \varepsilon E - Q = P$. Thus $t \varepsilon P - M = L_0$. We have shown that $t \varepsilon I_x^n$ implies that $t \varepsilon C^0 - K_n$ or $t \varepsilon L_0$, so $I_x^n \subseteq L_0 \cup (C^0 - K_n)$. It follows that (for each n)

$$L_0 \subseteq \Big(\bigcup_{x \varepsilon L_0} I_x^n\Big) \cap E \subseteq [L_0 \cup (C^0 - K_n)] \cap E.$$

Let $W_n = \bigcup_{x \varepsilon L_0} I_x^n$. By Lemma 6, W_n is equivalent to an open set.

$$\begin{aligned} L_0 &\subseteq \left(\bigcap_{n=1}^{\infty} W_n\right) \cap E \\ &\subseteq \left\{\bigcap_{n=1}^{\infty} [L_0 \cup (C^0 - K_n)]\right\} \cap E = \left\{L_0 \cup \bigcap_{n=1}^{\infty} (C^0 - K_n)\right\} \cap E \\ &= \{L_0 \cap E\} \cup \left\{\left[\bigcap_{n=1}^{\infty} (C^0 - K_n)\right] \cap E\right\} = L_0 \cup \phi = L_0 . \end{aligned}$$

Therefore $L_0 = (\bigcap_{n=1}^{\infty} W_n) \cap E$. Since each W_n is equivalent to an open set there exists a $G_0 \varepsilon \mathcal{G}_\delta$ such that

$$L_0 \sim G_0 \cap E.$$

Similar reasoning shows there exists a $G_1 \varepsilon \mathcal{G}_\delta$ such that

$$L_1 \sim G_1 \cap E.$$

Next we study the properties of M_0. It is convenient to define a function $\pi : R^2 \to R$ by $\pi(\langle x, y \rangle) = x$. If $x \varepsilon M \cap K_n$, then, starting at x and proceeding along γ_x, let $\sigma_n(x)$ be the first point of C_n^0 reached. Set $h_n^0(x) = \pi(\sigma_n(x))$ (for $x \varepsilon M \cap K_n$).

h_n^0 is an increasing function on $M \cap K_n$; for if $x_1, x_2 \varepsilon M \cap K_n$ and $x_1 < x_2$, then, since γ_{x_1} cannot meet γ_{x_2} in $\bar{A}_n^0$, it is evident (see Figure 2) that $\pi(\sigma_n(x_1)) < \pi(\sigma_n(x_2))$. (The argument can be rigorized by means of Theorem 11.8 on p. 119 in [6].) Since

$$\gamma_x \subseteq \{z \mid |z - x| \leqq 1\}, \qquad |x - h_n^0(x)| \leqq 1.$$

So by Lemma 7 h_n^0 can be extended to an increasing function h_n on C^0.

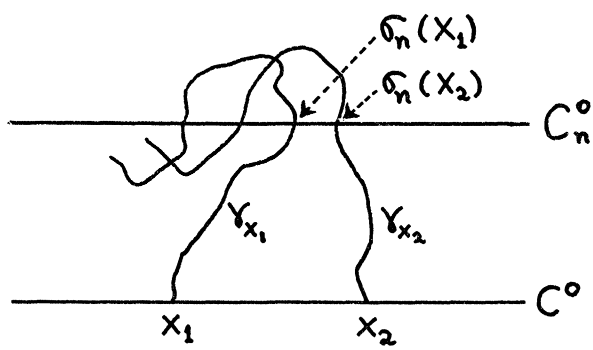

FIGURE 2.

Let

$$g_n(x) = f\left(\left\langle h_n(x), \frac{1}{n}\right\rangle\right).$$

For $x \in M \cap K_n$,

$$g_n(x) = f\left(\left\langle h_n^0(x), \frac{1}{n}\right\rangle\right) = f(\sigma_n(x)).$$

If $x \in M$, then for all sufficiently large n, $x \in M \cap K_n$, so

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(\sigma_n(x)) = \varphi(x).$$

Thus $g_n |_M \to \varphi |_M$. By III, $f(\langle x, 1/n\rangle)$ is a function (of x) of Baire class α, so by Lemma 8 we can choose, for each n, a countable set N_n such that $g_n |_{C^0 - N_n}$ is of Baire class α. Let $N = \bigcup_{n=1}^{\infty} N_n$. Then $g_n |_{M-N}$ is of Baire class α. But $g_n |_{M-N} \to \varphi |_{M-N}$, so $\varphi |_{M-N}$ is of Baire class $\alpha + 1$.

Now

$$P \cap (M - N) = (\varphi |_{M-N})^{-1}((-\infty, r]) = T \cap (M - N),$$

where $T \in \mathfrak{M}^{\alpha+1}$ (by IV and V). Clearly $P \cap M \sim T \cap M$.

We have

$$L = L_0 \cup L_1 \sim (G_0 \cap E) \cup (G_1 \cap E) = (G_0 \cup G_1) \cap E,$$

so $L \sim G \cap E$ where $G \in \mathcal{G}_\delta$. Also

$$M_0 = P \cap M \sim T \cap M = T \cap (E - L)$$
$$\sim T \cap [E - (G \cap E)] = [T \cap (C^0 - G)] \cap E.$$

Since $G \in \mathcal{G}_\delta$, $C^0 - G \in \mathcal{F}_\sigma$, so by VI and VIII, $T \cap (C^0 - G) \in \mathfrak{M}^{\alpha+1}$. Thus

$$M_0 \sim T_0 \cap E,$$

where $T_0 \in \mathfrak{M}^{\alpha+1}$. Now we can examine the properties of P.

$$P = L_0 \cup M_0 \sim (G_0 \cap E) \cup (T_0 \cap E) = (G_0 \cup T_0) \cap E,$$

so, again by VI and VIII,

$$P \sim T_1 \cap E,$$

where $T_1 \in \mathfrak{M}^{\alpha+1}$. Since a countable set is in $\mathcal{F}_\sigma$ and the complement of a countable set is in $\mathcal{G}_\delta$, it is easy to show (using VI and VIII) that

$$P = T_2 \cap E,$$

where $T_2 \in \mathfrak{M}^{\alpha+1}$. Since $P \cap Q = \phi$,

$$P \subseteq T_2 \subseteq C^0 - Q.$$

Remembering the definitions of P and Q, and observing the fact that $C^0 - \varphi^{-1}([t, +\infty)) = \varphi^{-1}((-\infty, t))$, we can summarize the results of the first part of the proof as follows.

For each pair r, t of real numbers with $r < t$, there exists a set $T(r, t) \in \mathfrak{M}^{\alpha+1}$ such that

$$\varphi^{-1}((-\infty, r]) \subseteq T(r, t) \subseteq \varphi^{-1}((-\infty, t)).$$

Given any real r, let $\{t_n\}$ be a strictly decreasing sequence of real numbers converging to r. Then

$$\varphi^{-1}((-\infty, r]) = \bigcap_{n=1}^{\infty} \varphi^{-1}((-\infty, t_n)).$$

So

$$\varphi^{-1}((-\infty, r]) \subseteq \bigcap_{n=1}^{\infty} T(r, t_n) \subseteq \bigcap_{n=1}^{\infty} \varphi^{-1}((-\infty, t_n)) = \varphi^{-1}((-\infty, r]),$$

and hence

$$\varphi^{-1}((-\infty, r]) = \bigcap_{n=1}^{\infty} T(r, t_n).$$

By VIII,

$$\varphi^{-1}((-\infty, r]) \in \mathfrak{M}^{\alpha+1}.$$

Since f is an arbitrary function of Baire class α in D^0 and φ is an arbitrary boundary function for f, we can replace f, φ, r by $-f$, $-\varphi$, $-r$ to find that

$$\varphi^{-1}([r, +\infty)) \in \mathfrak{M}^{\alpha+1}.$$

Also,

$$\varphi^{-1}((r, +\infty)) = C^0 - \varphi^{-1}((-\infty, r]) \in \mathfrak{M}^{\alpha+1}.$$

By IX, φ is of Baire class $\alpha + 1$. *Q.E.D.*

5. Boundary functions for measurable functions.

Theorem 5. *Let f be a real-valued Borel-measurable function in D^0 and let φ be a finite-valued boundary function for f. Then φ is Borel-measurable.*

Since every Borel-measurable function is of some Baire class α, this theorem is an immediate consequence of Theorem 4. We now show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable.

Let μ denote Lebesgue measure on R and let μ^2 denote Lebesgue measure on R^2. Let μ_e denote exterior Lebesgue measure on R; that is,

$$\mu_e(E) = \text{g.l.b. } \{\mu(G) \mid G \text{ is open and } E \subseteq G\},$$

for any set $E \subseteq R$.

Lemma 9. *Let h be an increasing real-valued function on a set $E \subseteq R$. Then there exists an open interval $I \supseteq E$ such that h can be extended to an increasing real-valued function on I.*

Proof. If E is unbounded below, set $a = -\infty$. If E is bounded below, set

$$a = \text{g.l.b. } E, \qquad \text{if} \quad (\text{g.l.b. } E) \notin E,$$

$$a = (\text{g.l.b. } E) - 1, \quad \text{if} \quad (\text{g.l.b. } E) \in E.$$

If E is unbounded above, set $b = +\infty$. If E is bounded above, set

$$b = \text{l.u.b. } E, \qquad \text{if} \quad (\text{l.u.b. } E) \notin E,$$

$$b = (\text{l.u.b. } E) + 1, \quad \text{if} \quad (\text{l.u.b. } E) \in E.$$

Let $I = (a, b)$. Clearly $E \subseteq I$. Let $e = \text{g.l.b. } E$ (e may be $-\infty$). For $x_0 \in (e, b)$ set

$$f(x_0) = \text{l.u.b. } \{h(x) \mid x \in (a, x_0] \cap E\}.$$

If $e = a$ we are done. If $e > a$ then $e \in E$. For $x_0 \in (a, e]$ set $f(x_0) = h(e)$. It is easily verified that f is finite-valued and increasing, and is an extension of h.

Lemma 10. *Let $E \subseteq R$ be a set of measure 0 and let h be an increasing function on E. Suppose $h(E)$ has measure 0. Then $\{x + h(x) \mid x \in E\}$ has measure 0.*

Proof. Extend h to an increasing function g on an open interval $I = (a, b) \supseteq E$. Set $g(a) = -\infty$ and $g(b) = +\infty$. Take any $\epsilon > 0$. Choose an open set G such that $I \supseteq G \supseteq E$ and $\mu(G) < \epsilon/2$. Choose an open set $H \supseteq h(E)$ with $\mu(H) < \epsilon/2$. Say

$$G = \bigcup_{n \in N} I_n, \quad \text{and} \quad H = \bigcup_{m \in M} J_m,$$

where $\{I_n \mid n \in N\}$ and $\{J_m \mid m \in M\}$ are countable families of disjoint open intervals. Let $I_n = (a_n, b_n)$, and observe that $a_n, b_n \in [a, b]$. Set

$$S = \bigcup_{n \in N} \{g(a_n), g(b_n)\} - \{-\infty, +\infty\}.$$

Notice that S is countable. Set

$$K_n = (g(a_n), g(b_n)).$$

One can easily verify that $k \neq n$ implies $K_k \cap K_n = \phi$.

If A and B are two subsets of R, let

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

It is easy to show that for any two intervals J and J', $\mu_e(J + J') \leqq \mu(J) + \mu(J')$.

Let $W = \{x + h(x) \mid x \in E\}$.

Assertion.

$$W \subseteq (E + S) \cup \bigcup_{n \in N} \bigcup_{m \in M} [(I_n \cap g^{-1}(J_m)) + (J_m \cap K_n)].$$

To prove this, let w be an arbitrary point of W. Write $w = x + h(x)$, where $x \in E$. For some n, $x \in I_n$. Since g is increasing,

$$h(x) = g(x) \in [g(a_n), g(b_n)].$$

If $h(x)$ equals $g(a_n)$ or $g(b_n)$, then $h(x) \in S$, so $w = x + h(x) \in E + S$. On the other hand, suppose $h(x) \neq g(a_n), g(b_n)$. Then $h(x) \in K_n$. Also, $g(x) = h(x) \in J_m$ for some m. Thus $h(x) \in J_m \cap K_n$ and $x \in I_n \cap g^{-1}(J_m)$, so that

$$w = x + h(x) \in (I_n \cap g^{-1}(J_m)) + (J_m \cap K_n).$$

This proves the Assertion.

Since g is increasing, $g^{-1}(J_m)$ is an interval, so both $I_n \cap g^{-1}(J_m)$ and $J_m \cap K_n$ are intervals. Also note that $m \neq l$ implies $g^{-1}(J_m) \cap g^{-1}(J_l) = \phi$. By the Assertion,

$$\begin{aligned}
\mu_e(W) &\leqq \mu_e(E + S) + \sum_{n \in N} \sum_{m \in M} \mu_e[(I_n \cap g^{-1}(J_m)) + (J_m \cap K_n)] \\
&\leqq \mu_e(E + S) + \sum_{n \in N} \sum_{m \in M} [\mu(I_n \cap g^{-1}(J_m)) + \mu(J_m \cap K_n)] \\
&= \mu_e\Big(\bigcup_{s \in S} (s + E)\Big) + \sum_{n \in N} \Big[\sum_{m \in M} \mu(I_n \cap g^{-1}(J_m)) + \sum_{m \in M} \mu(J_m \cap K_n)\Big] \\
&\leqq \sum_{s \in S} \mu_e(s + E) + \sum_{n \in N} \Big[\mu(I_n) + \sum_{m \in M} \mu(J_m \cap K_n)\Big] \\
&= 0 + \mu(G) + \sum_{n \in N} \sum_{m \in M} \mu(J_m \cap K_n) \\
&= \mu(G) + \sum_{m \in M} \sum_{n \in N} \mu(J_m \cap K_n) \\
&\leqq \mu(G) + \sum_{m \in M} \mu(J_m) = \mu(G) + \mu(H) < \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $\mu_e(W) = 0$.

Lemma 11. *Let $L = \{\langle x, a\rangle \mid x \in E\}$ and $M = \{\langle x, b\rangle \mid x \in R\}$ be two horizontal lines in R^2. Let E be a set of (linear) measure 0 in L and let F be a set of (linear) measure 0 in M. Let $\mathfrak{L}$ be a set of closed line segments such that*

(a) $s_1, s_2 \in \mathfrak{L}, s_1 \neq s_2 \Rightarrow s_1 \cap s_2 = \phi$

(b) $s \in \mathfrak{L} \Rightarrow$ *one endpoint of s lies in E and the other endpoint lies in F.*

Let $S = \bigcup_{k \in \mathfrak{L}} s$. Then $\mu^2(S) = 0$.

Proof. Assume without loss of generality that $b > a$. For any $\langle x, y\rangle \in R^2$ let $\pi(\langle x, y\rangle) = x$. For any $y \in R$ let $l_y = \{\langle x, y\rangle \mid x \in R\}$. Let

$$E_0 = \{z \in E \mid z \text{ is the endpoint of some } s \in \mathfrak{L}\},$$

and observe that E_0 has linear measure 0. For any set $A \subseteq R^2$ we of course set

$$\pi(A) = \{x \in R \mid \langle x, y\rangle \in A \quad \text{for some} \quad y \in R\}.$$

We define a function h on $\pi(E_0)$ as follows. If $x \in \pi(E_0)$, then $\langle x, a\rangle \in E_0$, so we can choose a (unique) segment $s \in \mathfrak{L}$ with one endpoint at $\langle x, a\rangle$. If the other endpoint of s is p, we set $h(x) = \pi(p)$. Clearly h maps $\pi(E_0)$ into $\pi(F)$.

Since the segments in $\mathcal{L}$ cannot intersect each other, h must be an increasing function.

Take any y_0 with $b > y_0 > a$. Let $c = b - y_0$, $d = y_0 - a$. A simple computation shows that if $q \in l_{y_0} \cap S$, then

$$\pi(q) = \frac{cx + dh(x)}{c + d},$$

for some $x \in \pi(E_0)$. So

$$\pi(l_{y_0} \cap S) \subseteq \left\{\frac{cx + dh(x)}{c + d} \,\middle|\, x \in \pi(E_0)\right\}.$$

Now $(d/c)h(x)$ is an increasing function mapping $\pi(E_0)$ into $(d/c)\pi(F)$, so by Lemma 10

$$\left\{x + \frac{d}{c}\, h(x) \,\middle|\, x \in \pi(E_0)\right\}$$

has measure 0. Hence

$$\frac{c}{c + d}\left\{x + \frac{d}{c}\, h(x) \,\middle|\, x \in \pi(E_0)\right\} = \left\{\frac{cx + dh(x)}{c + d} \,\middle|\, x \in \pi(E_0)\right\}$$

has measure 0, so $\mu(\pi(l_{y_0} \cap S)) = 0$. But $\mu(\pi(l_{y_0} \cap S)) = 0$ also when $y_0 \notin (a, b)$, so $\mu(\pi(l_y \cap S)) = 0$ for every y. If we knew that S were measurable, the lemma would follow immediately from the Fubini theorems. But since we have, as yet, no guaranty of the measurability of S, a more complicated argument is necessary. At several stages in the argument the reader will find it useful to draw diagrams to help him visualize the situation.

For any $y_1, y_2 \in R$, let

$$U(y_1, y_2) = \{\langle x, y\rangle \mid x, y \in R, y_1 < y < y_2\}.$$

A set of the form $U(y_1, y_2)$ will be referred to as a *horizontal open strip*.

For each positive integer n, let $\mathcal{L}(n)$ denote the set of all segments $s \in \mathcal{L}$ such that s has a point in common with $\{\langle x, b\rangle \mid x \in (-n, n)\}$. Let

$$S(n) = [\bigcup_{s \in \mathcal{L}(n)} s] \cap U\left(a + \frac{1}{n}, b - \frac{1}{n}\right).$$

Since l_a and l_b have (plane) measure 0, and since

$$S \subseteq l_a \cup l_b \cup \bigcup_{n=1}^{\infty} S(n),$$

it is sufficient to show that each $S(n)$ has measure 0.

Let n be a fixed positive integer. Set $a^* = a + 1/n$ and $b^* = b - 1/n$. Take any $\epsilon > 0$. Choose ϵ_0 so that $2\epsilon_0 + \epsilon_0^2 < \epsilon/(b - a)$. Let y_0 be any member of $[a^*, b^*]$. For the time being, y_0 will be held fixed.

For each $s \in \mathcal{L}$, let p_s be the endpoint of s on l_b, let q_s be the intersection point of s with l_{y_0}, and let r_s be the endpoint of s on l_a.

Choose an open set $G \subseteq R$ such that $\pi(l_{y_0} \cap S(n)) \subseteq G$ and $\mu(G) < \epsilon_0$. Say $G = \bigcup_i I_i$, where $I_i = (a_i, b_i)$ and the I_i's are pairwise disjoint. We may assume that each I_j contains a point of $\pi(l_{y_0} \cap S(n))$. For each j, let

$$c_j = g.l.b. \{\pi(p_s) \mid s \in \mathcal{L}(n), \quad \pi(q_s) \in I_j\},$$

$$d_j = l.u.b. \{\pi(p_s) \mid s \in \mathcal{L}(n), \quad \pi(q_s) \in I_j\},$$

$$c_j' = g.l.b. \{\pi(r_s) \mid s \in \mathcal{L}(n), \quad \pi(q_s) \in I_j\},$$

$$d_j' = l.u.b. \{\pi(r_s) \mid s \in \mathcal{L}(n), \quad \pi(q_s) \in I_j\}.$$

Note that $c_j \leqq d_j$ and $c_j' \leqq d_j'$. Since the segments in $\mathcal{L}$ cannot intersect each other, it is easily seen that the intervals (c_j, d_j) are all pairwise disjoint. It is also clear (from the definition of $\mathcal{L}(n)$) that each (c_j, d_j) is a subset of $(-n, n)$. Hence, if we set $\alpha_j = d_j - c_j$, we have $\sum_j \alpha_j \leqq 2n$.

For each j, let $s(j)$ be the line segment joining the two points $\langle c_j', a\rangle$, $\langle c_j, b\rangle$, and let $t(j)$ be the line segment joining the two points $\langle d_j', a\rangle$, $\langle d_j, b\rangle$. Let A_j be the closed subset of $\overline{U(a, b)}$ which is enclosed by the two line segments $s(j)$, $t(j)$. Let H_j denote the intersection of A_j with the horizontal open strip

$$V = U\left(\max\left\{a, y_0 - \frac{\epsilon_0}{2n^2}\right\}, \min\left\{b, y_0 + \frac{\epsilon_0}{2n^2}\right\}\right).$$

Note that H_j is measurable. Setting $H = \bigcup_j H_j$, it is clear from the definition of the A_j's that

$$S(n) \cap V \subseteq H.$$

Take any $y \in R$. We wish to show that

$$\mu(\pi(H \cap l_y)) < \frac{\epsilon}{b - a}.$$

We can, of course, assume that

$$y \in \left(\max\left\{a, y_0 - \frac{\epsilon_0}{2n^2}\right\}, \min\left\{b, y_0 + \frac{\epsilon_0}{2n^2}\right\}\right).$$

An elementary computation, using the geometrical properties of H_j, shows that

$$\mu(\pi(H_j \cap l_y)) \leqq \left(1 + \frac{|y - y_0|}{b - y_0}\right)\mu(I_j) + \alpha_j \frac{|y - y_0|}{b - y_0}.$$

Therefore

$$\begin{aligned}\mu(\pi(H \cap l_y)) &\leqq \sum_j \mu(\pi(H_j \cap l_y)) \\ &\leqq \left(1 + \frac{|y - y_0|}{b - y_0}\right) \sum_j \mu(I_j) + \frac{|y - y_0|}{b - y_0} \sum_j \alpha_j\end{aligned}$$

$$\leqq \left(1 + \frac{|y - y_0|}{1/n}\right)\mu(G) + \frac{|y - y_0|}{1/n} 2n$$

$$\leqq \left(1 + n \frac{\epsilon_0}{2n^2}\right)\epsilon_0 + \frac{\epsilon_0}{2n^2} 2n^2$$

$$\leqq 2\epsilon_0 + \epsilon_0^2 < \frac{\epsilon}{b - a},$$

so $\mu(\pi(H \cap l_y)) < \epsilon/(b - a)$ for every y.

We have shown that *for each* $y_0 \in [a^*, b^*]$ *there exists a horizontal open strip* $V(y_0)$ *containing* l_{y_0}, *and there exists a measurable set* $H(y_0) \subseteq V(y_0)$, *such that*

$$S(n) \cap V(y_0) \subseteq H(y_0)$$

and (for every y*)* $\pi(H(y_0) \cap l_y)$ *is measurable and*

$$\mu(\pi(H(y_0) \cap l_y)) < \frac{\epsilon}{b - a}.$$

The various open strips $V(y_0)$ ($y_0 \in [a^*, b^*]$) clearly cover the compact set $\{\langle 0, y\rangle \mid y \in [a^*, b^*]\}$. Choose a finite subcovering $V(y_1), V(y_2), \cdots, V(y_m)$. Set

$$K = \left[H(y_m) \cup \bigcup_{i=1}^{m-1}\left(H(y_i) - \bigcup_{j=i+1}^{m} V(y_j)\right)\right] \cap U(a^*, b^*).$$

Obviously K is measurable, and for each y, $\pi(K \cap l_y)$ is measurable and $\mu(\pi(K \cap l_y)) < \epsilon/(b - a)$. Moreover, $S(n) \subseteq K$. We have

$$\mu^2(K) = \int_{a^*}^{b^*} \mu(\pi(K \cap l_y))\, dy \leqq \int_{a^*}^{b^*} \frac{\epsilon}{b - a}\, dy = (b^* - a^*)\frac{\epsilon}{b - a} < \epsilon.$$

Since ϵ is arbitrary, this shows that

$$\text{g.l.b. } \{\mu^2(K) \mid K \text{ measurable}, \; S(n) \subseteq K\} = 0.$$

Therefore $S(n)$ has measure 0.

Lemma 12. *For every* $\epsilon > 0$ *there exists a strictly increasing function* h *on* R *such that* $h(R)$ *has measure* 0, *and for every* x, $|x - h(x)| \leqq \epsilon$.

Proof. For each (not necessarily positive) integer n, let $I_n = [n\epsilon, (n + 1)\epsilon]$. Then $\bigcup_n I_n = R$. There exists a strictly increasing function $f : [0, 1] \to [0, 1]$ such that $\mu(f([0, 1])) = 0$. For example, such a function may be defined as follows. Any number in [0, 1) may be written in the form

$$.a_1a_2a_3 \cdots a_n \cdots, \quad \text{(binary decimal)},$$

where the decimal does not end in an infinite unbroken string of 1's. Set

$$f(.a_1a_2a_3 \cdots a_n \cdots) = .b_1b_2b_3 \cdots b_n \cdots, \quad \text{(ternary decimal)},$$

where $b_i = 0$ if $a_i = 0$ and $b_i = 2$ if $a_i = 1$. Set $f(1) = 1$. f maps [0, 1] into the Cantor set, so $\mu(f([0, 1])) = 0$. It is easily shown that f is strictly increasing.

For each n, it is easy to obtain from f a function $f_n : I_n \to I_n$ such that f_n is strictly increasing and $\mu(f_n(I_n)) = 0$. Set

$$h(x) = f_n(x) \quad \text{for} \quad x \in (n\epsilon, (n + 1)\epsilon].$$

There is no difficulty in proving that h has the desired properties.

Theorem 6. *Let φ be an arbitrary function on $C^0 = \{\langle x, 0\rangle \mid x \in R\}$. Then there exists a function f on $D^0 = \{\langle x, y\rangle \mid y > 0\}$ such that $f(z) = 0$ almost everywhere and φ is a boundary function for f.*

Proof. For each positive integer n let h_n be a strictly increasing function on R such that $\mu(h_n(R)) = 0$, and for every x, $|x - h_n(x)| \leqq 1/n$. Let

$$E_n = \left\{\left\langle h_n(x), \frac{1}{n}\right\rangle \,\middle|\, x \in R\right\}.$$

E_n is a subset of

$$C_n^0 = \left\{\left\langle x, \frac{1}{n}\right\rangle \,\middle|\, x \in R\right\},$$

and E_n has linear measure 0. For each n, x let $s_n(x)$ be the line segment joining $\langle h_n(x), 1/n\rangle$ and $\langle h_{n+1}(x), 1/(n + 1)\rangle$. Since

$$h_n(x) > h_n(x') \Rightarrow x > x' \Rightarrow h_{n+1}(x) > h_{n+1}(x'),$$

we find that $x \neq x'$ implies $s_n(x) \cap s_n(x') = \phi$. Since each $s_n(x)$ has one endpoint in E_n and the other in E_{n+1}, Lemma 11 shows that for each n

$$\mu^2\Big(\bigcup_{x \in R} s_n(x)\Big) = 0.$$

Hence

$$\mu^2\left(\bigcup_{n=1}^{\infty} \bigcup_{x \in R} s_n(x)\right) = 0.$$

Set

$$f(z) = \varphi(\langle x, 0\rangle), \quad \text{if } z \in s_n(x) \text{ for some } n,$$
$$f(z) = 0, \quad \text{if } z \text{ is not in any } s_n(x).$$

$f(z) = 0$ almost everywhere. Let

$$\gamma(x) = \{\langle x, 0\rangle\} \cup \bigcup_{n=1}^{\infty} s_n(x).$$

Since the endpoints of $s_n(x)$ are at $\langle h_n(x), 1/n\rangle$ and $\langle h_{n+1}(x), 1/(n + 1)\rangle$, and since $\langle h_n(x), 1/n\rangle \to \langle x, 0\rangle$ as $n \to \infty$, it is clear that $\gamma(x)$ is an arc at $\langle x, 0\rangle$. Obviously

$$\lim_{\substack{z \to \langle x, 0\rangle \\ z \in \gamma(x)}} f(z) = \varphi(\langle x, 0\rangle).$$

This proves the theorem.

Corollary. *There exists a measurable function in D^0 having a nonmeasurable boundary function.*

6. Concluding remarks. Our theorem on boundary functions for continuous functions could have been proved by a small modification of the argument in Section 4, but the proof in Section 3 is shorter and neater.

The reader may wonder whether Theorem 4 holds true for functions taking values on the Riemann sphere as well as for real-valued functions. The theorem does, in fact, remain true in the sphere-valued case. If we regard the Riemann sphere Σ as a subset of R^3 and apply Theorem 4 to each component of f and φ, we find that φ is of Baire class $\alpha + 1$ with R^3 as the universal range space. It is then easy to show by means of Satz 2 in Banach's paper [3] that φ is of Baire class $\alpha + 1$ with Σ regarded as the universal range space. A similar procedure shows that Theorem 5 also remains true for functions taking values on the Riemann sphere.

The results of Sections 2, 3 and 4 cannot be extended to three dimensions—at least not in the most obvious way. We can show this as follows. Let K be an open cube in R^3 and let F be one face of K. If f is defined in K, then we say φ (defined on F) is a *boundary function* for f provided that for each $x \in F$ there exists an arc γ with one endpoint at x such that $\gamma - \{x\} \subseteq K$ and

$$\lim_{\substack{v \to x \\ v \in \gamma}} f(v) = \varphi(x).$$

Lemma 13. *Suppose that every point of F is an ambiguous point of the function $f : K \to R^3$. Then f has a nonmeasurable boundary function.*

Proof. Let E be a nonmeasurable subset of F. Since each point of F is an ambiguous point we can choose, for each $x \in F$, two distinct points $\varphi_1(x), \varphi_2(x) \in R^3$ such that there exist arcs γ_i at x with

$$\lim_{\substack{v \to x \\ v \in \gamma_i}} f(v) = \varphi_i(x), \qquad (i = 1, 2).$$

Let

$$\varphi(x) = \varphi_1(x), \quad \text{if} \quad x \in E,$$
$$\varphi(x) = \varphi_2(x), \quad \text{if} \quad x \in F - E.$$

Then

$$\varphi(x) - \varphi_1(x) = 0, \quad \text{if} \quad x \in E,$$
$$\varphi(x) - \varphi_1(x) \neq 0, \quad \text{if} \quad x \in F - E.$$

Therefore $(\varphi - \varphi_1)^{-1}(\{0\}) = E$, so $\varphi - \varphi_1$ is not a measurable function. Hence either φ or φ_1 is a nonmeasurable function. Since φ and φ_1 are both boundary functions for f, the lemma is proved.

P. T. Church [4] has constructed an example of a homeomorphism f from K onto K such that every point of F is an ambiguous point for f. By Lemma 13, f has a nonmeasurable boundary function φ. Theorem 1 is therefore false in three dimensions. Write f and φ in terms of their components; say $f = \langle f_1, f_2, f_3 \rangle$ and $\varphi = \langle \varphi_1, \varphi_2, \varphi_3 \rangle$. Since φ is nonmeasurable, one of its components, say φ_i, is nonmeasurable. But φ_i is a boundary function for the continuous real-valued function f_i, so Theorem 2 and Theorem 4 must be false in three dimensions.

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